The Method of Characteristics Revisited
A Viability Approach

A Mini-Course[1]

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Contents

Introduction

This mini-course provides a presentation of the method of characteristics to initial/boundary-value problems for systems of first-order partial differential equations and to Hamilton-Jacobi variational inequalities.

These results are indeed useful for the treatment of hybrid systems of control theory.

We use the tools forged by set-valued analysis and viability theory, which happen to be both efficient and versatile to cover many problems. They find here an unexpected relevance.

Indeed, since solutions to first-order systems

$$\forall\, j = 1, \ldots, p, \quad \frac{\partial}{\partial t} u(t,x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} u_i(t,x) f_i(t,x,u(t,x)) - g_j(t,x,u(t,x)) \;=\; 0$$

may have "shocks", i.e., may be set-valued maps (or multi-valued maps), it seems to us natural to use the concept of "graphical derivative" of a set-valued map from set-valued analysis instead of "distributional derivative" from the theory of distributions to give a meaning to a concept of solution to such systems of partial differential equations.

The basic concept useful in our framework is the concept of *capture basin* of a subset C under a differential equation, which is the set of points from which a solution to the differential equation reaches C in finite time.

Then we shall prove that *the graph of the solution $(t,x) \rightsquigarrow U(t,x)$ to the above boundary value problem is the capture basin of the graph of the initial/boundary data under the characteristic system of differential equations*

$$\begin{cases} i) & \tau'(t) \;=\; -1 \\ ii) & x'(t) \;=\; -f(\tau(t), x(t), y(t))) \\ iii) & y'(t) \;=\; -g(\tau(t), x(t), y(t)) \end{cases}$$

and that, under adequate assumptions, this solution is **unique** *among the solutions with closed graph to this boundary value problem.*

Such a solution is thus taken in a generalized — or weak — sense (Frankowska solutions), since such maps, even when they are single-valued, are not differentiable in the usual sense. But when they are, they naturally coincide with the above concept of solution thanks to the uniqueness property.

Existence and uniqueness of the solution is obtained

1. from a first characterization of the capture basin of C as the union of reachable sets $\vartheta_{-f}(t, C) := \{\vartheta_{-f}(t, c)\}_{c \in C}$, where $\vartheta_{-f}(t, c)$ denotes the value at time t of the solution to the differential equation $x' = -f(x)$ starting at c, allowing explicit computations of the capture basins in specific instances,

2. from a second characterization of the capture basin of C of a closed backward invariant subset M which is a repeller[1] under a differential equation $x' = f(x)$. It states that the capture basin is the **unique** closed subset K such that

   (a) $C \subset K \subset M$,

   (b) K is *backward invariant* in the sense that any backward solution starting from K is viable in K,

   (c) $K \backslash C$ is *locally viable* in the sense that for any $x \in K \backslash C$, there exist $T > 0$ and a solution to the differential equation starting at x viable in $K \backslash C$ on the interval $[0, T]$,

3. from the 1942 Nagumo Theorem stating that

   (a) the subset K is backward invariant under a differential inclusion $x' = f(x)$ if and only if, for every $x \in K$, $f(x) \in -T_K(x)$, or equivalently, if and only if, for every $x \in K$, $p \in N_K(x)$, $\langle p, f(x) \rangle \geq 0$,

   (b) the subset $K \backslash C$ is locally viable if and only if, for every $x \in K \backslash C$, $f(x) \in T_K(x)$, or equivalently, if and only if, for every $x \in K \backslash C$, $p \in N_K(x)$, $\langle p, f(x) \rangle \leq 0$,

   where the "contingent cone" $T_K(x)$ to a subset K at a point $x \in K$, introduced in the early thirties independently by Bouligand and Severi, adapts to any subset the concept of tangent space to manifolds, and where the "normal cone" $N_K(x) := T_K(x)^-$, defined as the polar cone to the contingent cone, adapts to any subset the concept of normal space to manifolds,

4. from the definition of the contingent derivative $DU(t, x, y)$ of the set-valued map $U : (t, x) \rightsquigarrow U(t, x)$ at a point (t, x, y) of the graph of U as the set-valued map from $\mathbf{R} \times X$ to Y the graph of which is the contingent cone to the graph of U at the point (t, x, y).

The above results — which are interesting by themselves for other mathematical models of evolutionary economics, population dynamics, epidemiology — can be applied to many other problems. Dealing with subsets, they can be applied to graphs

[1] This means that all solutions to the differential equation $x' = f(x)$ starting from M leave M in finite time.

of single-valued maps as well as set-valued maps, to epigraphs and hypographs of (extended) real-valued functions, to graph of "impulse" maps (which take empty values except in a discrete sets, useful in the study of hybrid systems or inventory management), etc. Since these results are also valid for underlying differential inclusions, we are able to treat control problems for such boundary-value problems for systems of first-order partial differential equations.

We also illustrate the strategy of using the properties of the viability kernel — the subset of elements from which starts at least a solution to the differential equation viable in K — and the capture basin of a subset advocated by Hlne Frankowska[2] for characterizing the value functions of some variational problems or stopping time problems as "contingent solutions" and/or "viscosity solutions to Hamilton-Jacobi "differential variational inequalities"

$$\begin{cases} i) & \mathbf{u}(x) \ \le \ \mathbf{u}^{\top}(x) \\ ii) & \left\langle \frac{\partial}{\partial x}\mathbf{u}^{\top}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\top}(x) \ \le \ 0 \\ iii) & (\mathbf{u}(x) - \mathbf{u}^{\top}(x)) \left(\left\langle \frac{\partial}{\partial x}\mathbf{u}^{\top}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\top}(x) \right) \ = \ 0 \end{cases}$$

and

$$\begin{cases} i) & 0 \ \le \ \mathbf{u}^{\perp}(x) \ \le \ \mathbf{u}(x) \\ ii) & \left\langle \frac{\partial}{\partial x}\mathbf{u}^{\perp}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\perp}(x) \ \ge \ 0 \\ iii) & (\mathbf{u}(x) - \mathbf{u}^{\perp}(x)) \left(\left\langle \frac{\partial}{\partial x}\mathbf{u}^{\perp}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\perp}(x) \right) \ = \ 0 \end{cases}$$

where

1. $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ defines the dynamics of the differential equation $x' = f(x)$,

2. $l : (x, p) \in \mathbf{R}^n \times \mathbf{R}^n \mapsto l(x, p) \in \mathbf{R}_+$ is a nonnegative "Lagrangian",

[2] Hélène Frankowska proved that the epigraph of the value function of an optimal control problem — assumed to be only lower semicontinuous — is invariant and backward viable under a (natural) auxiliary system. Furthermore, when it is continuous, she proved that its epigraph is viable and its hypograph invariant ([32, 33, 35, Frankowska]). By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton-Jacobi equation in the sense of M. Crandall and P.-L. Lions. See also [19, Barron & Jensen], [17, Barles] and [16, Bardi & Capuzzo-Dolcetta] for more details. Such concepts have been extended to solutions of systems of first-order partial differential equation without boundary conditions by Hélène Frankowska and the author (see [9, 10, 11, 12, 14, Aubin & Frankowska] and chapter 8 of [2, Aubin]). See also [6, 7, Aubin & Da Prato]. This point of view is used here in the case of boundary value problems.

3. $\mathbf{u} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is an extended nonnegative function (regarded as an obstacle, as in unilateral mechanics).

We shall observe that the epigraphs of the solutions are respectively the viability kernel and the capture basin of the epigraph of $\mathbf{u}$ under the map $g : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n \times \mathbf{R}$ defined by

$$g(x, y) \ := \ (f(x), -ay - l(x, f(x)))$$

This allows us to compute the "solutions" to these Hamilton-Jacobi equations and to check that they are respectively defined by

$$\mathbf{u}^{\top}(x) \ := \ \sup_{t \geq 0} \left(e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau)) d\tau \right)$$

and

$$\mathbf{u}^{\perp}(x) \ := \ \inf_{t \geq 0} \left(e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau)) d\tau \right)$$

where $x(\cdot)$ is the solution to the differential equation $x' = f(x)$ starting at x.

Using the fact that the contingent cone to the epigraph of an extended lower semicontinuous function is the epigraph of its "contingent epiderivative", the characterizations of viability kernel and capture basins in terms of tangential conditions allows to interpret these "solutions" as "contingent solutions" to the above Hamilton-Jacobi variational inequalities. Using the characterization in terms of normal cones, we obtain the equivalent interpretation of these solutions as viscosity solutions whenever the solution is continuous instead of being merely lower semicontinuous.

This this is the reason why this mini course begins with the Nagumo Theorem on closed subsets viable and/or invariant under a differential equation. Next, the notions of the "capture basin" of a closed subset, as well as its "viability kernel" are characterized and linked with the Nagumo Theorem. After recalling the concepts of contingent epiderivative and subdifferential of an extended function, these theorems are used for solving some Hamilton-Jacobi variational inequalities. They are also used for solving initial/boundary value problems for systems of first-order partial differential equations.

For the sake of simplicity, we restrict ourselves in this introductory mini-course to the simple case when the characteristic system is made of differential equations. However, the very same methods can be adapted to the case when the characteristic system is made of differential inclusions. Actually, the methods and the view points developed in these notes are more important than the examples presented here, since they can be easily generalized and efficiently used for solving many more difficult problems.

Chapter 1

The Nagumo Theorem

Introduction

This chapter is a presentation of the basic Nagumo Theorem and its corollaries and consequence in the simple framework of ordinary differential equations $x' = f(x)$. These results — which are interesting by themselves as mathematical metaphors of evolutionary economics, population dynamics, epidemiology, biological evolution when they are extended to differential inclusions — can be applied to many other problems, such as control problems and, as we shall illustrate here, can be used as tools for solving other mathematical problems. Dealing with subsets, they can be applied to graphs of single-valued maps as well as set-valued maps, to epigraphs and hypographs of (extended) real-valued functions, and to be used as versatile and efficient tools for solving systems of first-order partial differential equations.

A function $[0, T] \ni t \to x(t)$ is said to be (locally) *viable* in a given subset K on $[0, T]$ if, for any $t \in [0, T]$, the state $x(t)$ remains in K.

Therefore, if a continuous map $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ describes the dynamics of the system

$$\forall\, t \geq 0, \;\; x'(t) \; = \; f(x(t))$$

we shall say that K is *viable under* f if starting from any initial point of K, *at least one solution* to the differential equation is viable in K.

The Nagumo Theorem characterizes such a viability property for any locally compact subset K in terms of contingent and normal cones.

Hence, we begin by recalling the concept of contingent cones and normal cones to a arbitrary subset of a finite dimensional vector space. For that purpose, we start with the notion of upper convergence of sets introduced by Painlevé: The upper limit of a sequence of subsets $K_n \subset X$ is the set of cluster points of sequences $x_n \in K_n$.

With that concept, we can define **the contingent cone** $T_K(x)$ to a subset K at x is the upper limit of the "difference quotients" $\frac{K-x}{h}$. The normal cone $N_K(x)$ is next defined as the polar cone $T_K(x)^-$ to the contingent cone.

The Nagumo Theorem characterizes such a viability property for any locally compact subset K by stating that K is viable under f if and only if

$$\forall\, x \in K, \;\; f(x) \;\in\; T_K(x)$$

or, equivalently, if and only if

$$\forall\, x \in K, \; \forall\, p \in N_K(x) \;\; \langle p, f(x)\rangle \;\leq\; 0$$

Since open subsets, closed subsets, intersection of open and closed subsets of $\mathbf{R}^n$ are locally compact, we shall be able to specify this theorem in each of theses cases.

We then state and prove these viability theorems. Many proofs of viability theorems are now available: We chose the most elementary one (which is not the shortest) because it is the prototype of the extensions of the viability theorems. It is just a modification of the Euler method of approximating a solution by piecewise linear functions (polygonal lines) in order to force the solution to remain viable in K.

1.1 Viability & Invariance Properties

Definition 1.1.1 (Viable functions) *Let K be a subset of a finite dimensional vector-space $\mathbf{R}^n$. We shall say that a function $x(\cdot)$ from $[0,T]$ to $\mathbf{R}^n$ is* viable in K on $[0,T]$ *if for all $t \in [0,T]$, $x(t) \in K$.*

Let us describe the dynamics of the system by a map f from $\mathbf{R}^n$ to $\mathbf{R}^n$. We consider the initial value problem (or Cauchy problem) associated with the differential equation

$$\forall\, t \in [0,T], \;\; x'(t) \;=\; f(x(t)) \tag{1.1}$$

satisfying the initial condition $x(0) = x_0$.

Definition 1.1.2 (Viability & Invariance Properties) *Let $K \subset \mathbf{R}^n$. We shall say that K* is locally viable under f *if for any initial state x_0 of K,* there exist *$T > 0$ and a* viable *solution on $[0,T]$ to differential equation (1.1) starting at x_0. It is said to be (globally) viable under f if we can always take $T = \infty$.*

The subset K is said to be locally invariant under f *if for any initial state x_0 of K and for* all *solutions $x(\cdot)$ to differential equation (1.1) (a priori defined on $\mathbf{R}^n$)*

starting from x_0, there exists $T > 0$ such that $x(\cdot)$ is viable in K *on* $[0, T]$. *It is said to be (globally) invariant under f if we can always take $T = \infty$ for all solutions.*

A subset K is a repeller *if from any initial element $x_0 \in K$, all solutions to the differential equation (1.1) starting at $x_0 \in K$ leave K in finite time.*

Remark — We should emphasize that the concept of invariance *depends upon the behavior of f on the domain $\mathbf{R}^n$ outside K.* But we observe that viability property depends only on the behavior of f on K. □

So, the viability property requires only the existence of at least one viable solution whereas the invariance property demands that all solutions, if any, are viable.

Observe also that whenever there exists a unique solution to differential equation $x' = f(x)$ starting from any initial state x_0, then viability and invariance properties of a closed subset K are naturally equivalent.

We shall begin by characterizing the subsets K which are viable under f. The idea is simple, intuitive and makes good sense: *A subset K is viable under f if at each state x of K, the velocity $f(x)$ is "tangent" to K at x, so to speak, for bringing back a solution to the differential equation inside K.*

1.2 Contingent and Normal Cones

Limits of Sets

Limits of sets have been introduced by Paul Painlevé in 1902 without the concept of topology. They have been popularized by Kuratowski in his famous book Topologie and thus, often called *Kuratowski lower and upper limits* of sequences of sets. They are defined without the concept of a topology on the power space.

Definition 1.2.1 *Let $(K_n)_{n\in\mathbf{N}}$ be a sequence of subsets of a finite dimensional vector space X. We say that the subset*

$$\mathrm{Limsup}_{n\to\infty} K_n \ := \ \left\{ x \in X \ \mid \ \liminf_{n\to\infty} d(x, K_n) = 0 \right\}$$

is the upper limit *or* outer limit[1] *of the sequence K_n and that the subset*

$$\mathrm{Liminf}_{n\to\infty} K_n \ := \ \{ x \in X \ \mid \ \lim_{n\to\infty} d(x, K_n) = 0 \}$$

is its lower limit *or* inner limit. *A subset K is said to be the* limit *or the* set limit *of the sequence K_n if*

$$K \ = \ \mathrm{Liminf}_{n\to\infty} K_n \ = \ \mathrm{Limsup}_{n\to\infty} K_n \ =: \ \mathrm{Lim}_{n\to\infty} K_n$$

[1]The terms outer and inner limits of sets have been proposed by R.T Rockafellar and R. Wets.

Lower and upper limits are obviously *closed*. We also see at once that

$$\mathrm{Liminf}_{n\to\infty} K_n \ \subset\ \mathrm{Limsup}_{n\to\infty} K_n$$

and that the upper limits and lower limits of the subsets K_n and of their closures $\overline{K}_n$ do coincide, since $d(x, K_n) = d(x, \overline{K}_n)$.

Any decreasing sequence of subsets K_n has a limit:

$$\text{if } \ K_n \subset K_m \ \text{ when } \ n \geq m, \ \text{ then } \ \mathrm{Lim}_{n\to\infty} K_n \ = \ \bigcap_{n\geq 0} \overline{K_n}$$

An upper limit may be empty (no subsequence of elements $x_n \in K_n$ has a cluster point.)

Concerning sequences of singletons $\{x_n\}$, the set limit, when it exists, is either empty (the sequence of elements x_n is not converging), or is a singleton made of the limit of the sequence.

It is easy to check that:

Proposition 1.2.2 *If* $(K_n)_{n\in\mathbf{N}}$ *is a sequence of subsets of a finite dimensional vector space, then* $\mathrm{Liminf}_{n\to\infty} K_n$ *is the* set of limits of sequences $x_n \in K_n$ *and* $\mathrm{Limsup}_{n\to\infty} K_n$ *is the set* of cluster points of sequences $x_n \in K_n$, *i.e., of limits of subsequences* $x_{n'} \in K_{n'}$.

Contingent Cone

We reformulate the definition of contingent direction to a subset of a finite dimensional vector space introduced independently by Georges Bouligand and Francesco Severi [2] in the 30's:

Definition 1.2.3 *When* $K \subset X$ *is a subset of a normed vector space* X *and when* $x \in K$, *the set* $T_K(x)$

$$T_K(x) \ := \ \mathrm{Limsup}_{h\to 0+} \frac{K - x}{h}$$

of contingent directions *to* K *at* x *is a closed cone, called the* contingent cone *or simply, the* tangent cone, *to* K *at* x.

[2] The concepts of *semitangenti* and of *corde improprie* to a set at a point of its closure had been introduced by the Italian geometer Francesco Severi (1879-1961) and are equivalent to the concepts of *contingentes* and *paratingentes* introduced independently by the French mathematician Georges Bouligand, slightly later. Severi explains for the second time that he had discovered these concepts developed by Bouligand in *" suo interessante libro recente"* and comments: *"All'egregio geometra è evidentemente sfuggito che le sue ricerche in proposito sono state iniziate un po' più tardi delle mie ... Ma non gli muovo rimprovero per questo, perché neppur io riesco a seguire con cura minuziosa la bibliografica e leggo più volontieri una memoria o un libro dopo aver pensato per conto mio all'argomento."* I am grateful to M. Bardi for this information about Severi.

Therefore, a direction $v \in X$ is *contingent* to K at x if and only if

$$\liminf_{h \to 0+} \frac{d(x+hv, K)}{h} \ = \ 0$$

or, equivalently, if and only if there exists a sequence of elements $h_n > 0$ converging to 0 and a sequence of $v_n \in X$ converging to v such that

$$\forall \, n \geq 0, \ \ x + h_n v_n \ \in \ K$$

The lemma below shows right away why these cones will play a crucial role: they appear naturally whenever we wish to differentiate viable functions.

Lemma 1.2.4 *Let $x(\cdot)$ be a differentiable* viable *function from $[0, T]$ to K. Then*

$$\forall \, t \in [0, T[, \quad x'(t) \ \in \ T_K(x(t))$$

Proof — Let us consider a function $x(\cdot)$ viable in K. It is easy to check that $x'(0)$ belongs to the contingent cone $T_K(x_0)$ because $x(h)$ belongs to K and consequently,

$$\frac{d_K(x_0 + hx'(0))}{h} \ \leq \ \frac{\|x(0) + hx'(0) - x(h)\|}{h} \text{ converges to } 0$$

Hence $x'(0)$ belongs to the contingent cone to K at x_0. $\square$

For convex subsets K, the contingent cone coincides with the closed cone spanned by $K - x$:

Proposition 1.2.5 *Let us assume that K is convex. Then the contingent cone $T_K(x)$ to K at x is convex and*

$$T_K(x) \ = \ \overline{\bigcup_{h>0} \frac{K-x}{h}}$$

We shall say in this case that it is the tangent cone *to the convex subset K at x.*

Proof — We begin by stating the following consequence of convexity: If $0 < h_1 \leq h_2$, then

$$\frac{K-x}{h_2} \ \subset \ \frac{K-x}{h_1}$$

because $x + h_1 v = \frac{h_1}{h_2}(x + h_2 v) + \left(1 - \frac{h_1}{h_2}\right) x$ belongs to K whenever $x + h_2 v$ belongs to K. The sequence of the subsets $\frac{K-x}{h}$ being increasing, our proposition ensues. $\square$

Definition 1.2.6 (Viability Domain) *Let K be a subset of $\mathbf{R}^n$. We shall say that K is a* viability domain *of the map $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ if*

$$\forall\, x \in K, \;\; f(x) \;\in\; T_K(x) \tag{1.2}$$

We recall that a subset $K \subset \mathbf{R}^n$ is *locally compact* if there exists $r > 0$ such that the ball $B_K(x_0, r) := K \cap (x_0 + rB)$ is *compact*. Closed subsets, open subsets and intersections of closed and open subsets of a finite dimensional vector space are locally compact.

Normals

Definition 1.2.7 *The polar cone $N_K(x) := (T_K(x))^-$ is called the* normal cone *to K at x.*

It is also called the *Bouligand normal cone*, or the *contingent normal cone*, or also, the *sub-normal cone* and more recently, the *regular normal cone* by R.T. Rockafellar and R. Wets. In this book, only (regular) normals are used, so that we shall drop the adjective "regular". □

Lemma 1.2.8 *Let $K \subset X$ be a closed subset. Let $y \notin K$ and $x \in \Pi_K(y)$ a best approximation of y by elements of K: $\|y - x\| = d(y, K)$. Then*

$$\forall\, y \notin K, \; \forall\, x \in \Pi_K(y), \quad y - x \;\in\; N_K(x)$$

Proof — Since $x \in \Pi_K(y)$ minimizes the distance $z \mapsto \frac{1}{2}\|y - z\|^2$ on K, we deduce that

$$\forall\, v \in T_K(x), \;\; 0 \;\leq\; \langle x - y, v\rangle$$

so that $y - x$ belongs to $T_K(x)^- =: N_K(x)$. □

When K is convex, we deduce that $N_K(x)$ is the polar cone to $K - x$ because the tangent cone is spanned by $K - x$:

Theorem 1.2.9 *Let K be a closed convex subset of a finite dimensional vector space. Then*

$$p \in N_K(x) \;\; \textit{if and only if} \;\; \forall\, y \in K, \;\; \langle p, y\rangle \;\leq\; \langle p, x\rangle$$

and the graph of the set-valued map $x \rightsquigarrow N_K(x)$ is closed in $X \times X^\star$.

Proof — Let us consider sequences of elements $x_n \in K$ converging to x and $p_n \in N_K(x_n)$ converging to p. Then inequalities

$$\forall\, y \in K, \;\; \langle p_n, y\rangle \;\le\; \langle p_n, x_n\rangle$$

imply by passing to the limit inequalities

$$\forall\, y \in K, \;\; \langle p, y\rangle \;\le\; \langle p, x\rangle$$

which state that p belongs to $N_K(x)$. Hence the graph is closed. $\Box$

In the general case, we provide now the following characterization of the normal cone:

Proposition 1.2.10 *Let K be a subset of a finite dimensional vector-space X. Then $p \in N_K(x)$ if and only*

$$\left\{ \begin{array}{l} \forall\, \varepsilon > 0, \; \exists\, \eta > 0 \;\text{ such that }\; \forall\, y \in K \cap B(x,\eta), \\ \\ \langle p, y - x\rangle \;\le\; \varepsilon \|y - x\| \end{array} \right. \tag{1.3}$$

Proof — Let p satisfy above property (1.3) and $v \in T_K(x)$. Then there exist h_n converging to 0 and v_n converging to v such that

$$y \;:=\; x + h_n v_n \;\in\; K \cap B(x,\eta)$$

for n large enough. Consequently, inequalities $\langle p, v_n\rangle \le \varepsilon$ imply by taking the limit that $\langle p, v\rangle \le \varepsilon$ for all $\varepsilon > 0$. Hence $\langle p, v\rangle \le 0$, so that any element p satisfying the above property belongs to the polar cone of $T_K(x)$.

Conversely, assume that p violates property (1.3): There exist $\varepsilon > 0$ and a sequence of elements $x_n \in K$ converging to x such that

$$\langle p, x_n - x\rangle \;>\; \varepsilon \|x_n - x\|$$

We set $h_n := \|x_n - x\|$, which converges to 0, and $v_n := (x_n - x)/h_n$. These elements belonging to the unit sphere, a subsequence (again denoted) v_n converges to some v. By definition, this limit belongs to $T_K(x)$, so that $\langle p, v\rangle \le 0$. But our choice implies that $\langle p, v_n\rangle > \varepsilon$, so that $\langle p, v\rangle \ge \varepsilon$, a contradiction. $\Box$

We provide a useful characterization by duality of viability domains in terms of normal cones:

Theorem 1.2.11 *Let K be a locally compact subset of a finite dimensional vector space $\mathbf{R}^n$ and $f : K \mapsto \mathbf{R}^n$ be a continuous single-valued map. Then*

$$\forall\, x \in K, \;\; f(x) \;\in\; T_K(x) \tag{1.4}$$

if and only if

$$\forall\, x \in K, \quad f(x) \ \in \ \overline{\mathrm{co}}(T_K(x)) \tag{1.5}$$

or, equivalently, in terms of normal cone, if and only if

$$\forall\, x \in K,\ \forall\, p \in N_K(x), \quad \langle p, f(x)\rangle \ \leq \ 0$$

Proof — Since the normal cone $N_K(x)$ is the polar cone to $T_K(x)$, and thus, to $\overline{\mathrm{co}}(T_K(x))$, then $\overline{\mathrm{co}}(T_K(x))$ is the polar cone to $N_K(x)$, so that the two last statements are equivalent by polarity. Since the first statement implies the second one, it remains to prove that if for any $x \in K$, $f(x)$ belongs to $\overline{\mathrm{co}}(T_K(x))$, then $f(x)$ is actually contingent to K at x for any $x \in K$. This follows from the following

Lemma 1.2.12 *Let $K \subset \mathbf{R}^n$ be a locally compact subset of a finite dimensional vector space $\mathbf{R}^n$ and $f : K \mapsto \mathbf{R}^n$ be a continuous single-valued map. Assume that there exists $\alpha > 0$ such that*

$$\forall\, x \in K \cap B(x_0, \alpha), \quad f(x) \ \in \ \overline{\mathrm{co}}(T_K(x))$$

Then, $f(x)$ is contingent to K at elements x of a neighborhood of x_0. Actually, for any $\varepsilon > 0$, there exists $\eta(x_0, \varepsilon) \in]0, \alpha]$ such that

$$\forall\, x \in B(x_0, \eta(x_0,\varepsilon)),\ \forall\, h \leq \eta(x_0,\varepsilon), \quad d\left(f(x), \frac{\Pi_K(x + hf(x)) - x}{h}\right) \ \leq \ \varepsilon \tag{1.6}$$

where $\Pi_K(y)$ denotes the set of best approximations $\bar{x} \in K$ of y, i.e., the solutions to $\|\bar{x} - y\| = d(y, K)$.

Proof — It is sufficient to check the Lemma when $f(x) \neq 0$. Let us set

$$g(t) \ := \ \frac{1}{2} d(x + tf(x), K)^2 \ = \ \|x + tf(x) - x_t\|^2$$

where $x_t \in \Pi_K(x+tf(x))$ is a best approximation of $x+tf(x)$ by elements of K. We take α small enough for $K \cap B(x_0, \alpha)$ to be compact. We observe that there exists $\beta \in]0, \alpha]$ such that for all $x \in B(x_0, \beta)$, $\|f(x)\| \leq 2\|f(x_0)\|$ because f is continuous at x_0. Furthermore,

$$\|x + tf(x) - x_t\| \ = \ d(x + tf(x), K) \ \leq \ t\|f(x)\| \ \leq \ 2t\|f(x_0)\|$$

because x belongs to $K \cap B(x_0, \beta)$, so that $\|x - x_t\| \ \leq \ 2t\|f(x)\| \ \leq \ 4t\|f(x_0)\|$ converges to 0 with t.

On the other hand, for every $v_t \in T_K(x_t)$, there exists a sequence of $h_n > 0$ converging to 0 and v_t^n converging to v_t such that $x_t + h_n v_t^n$ belongs to K. Therefore,

$$g(t + h_n) - g(t) \ \leq \ \frac{1}{2}\left(\|x + tf(x) - x_t + h_n(f(x) - v_t^n)\|^2 - \|x + tf(x) - x_t\|^2\right)$$

and thus, dividing by $h_n > 0$ and letting h_n converge to 0, that

$$\forall\, v_t \in T_K(x_t),\;\; g'(t) \;\leq\; \langle x + tf(x) - x_t, f(x) - v_t\rangle$$

Since it is true for any $v_t \in T_K(x_t)$ and since the right-hand side is affine with respect to v_t, we deduce that this inequality remains true for any $v_t \in \overline{\text{co}}(T_K(x_t))$, and thus, by assumption, for $f(x_t) \in \overline{\text{co}}(T_K(x_t))$:

$$g'(t) \;\leq\; \langle x + tf(x) - x_t, f(x) - f(x_t)\rangle \leq 2t\|f(x_0)\|(\|f(x) - f(x_0)\| + \|f(x_0) - f(x_t)\|)$$

For any ε, let $\gamma > 0$ such that $\|f(y) - f(x_0)\| \leq \frac{\varepsilon^2}{8\|f(x_0)\|}$ whenever $y \in B(x_0, 2\gamma)$. Since $\|x_0 - x_t\| \leq \|x - x_0\| + 4t\|f(x_0)\| \leq 2\gamma$ whenever $\|x - x_0\| \leq \gamma$ and $t \leq \frac{\gamma}{4\|f(x_0)\|}$, then, setting $\eta(x_0, \varepsilon) := \min\left(\gamma, \frac{\gamma}{4\|f(x_0)\|}\right)$, we obtain

$$\forall\, x \in K \cap B(x_0, \eta(x_0, \varepsilon)),\; \forall\, t \in]0, \eta(x_0, \varepsilon)],\;\; g'(t) \;\leq\; \frac{t}{2}\varepsilon^2 \tag{1.7}$$

Therefore, after integration from 0 to h, we obtain

$$\forall\, h \in [0, \eta(x_0, \varepsilon)],\;\; g(h) - g(0) \;\leq\; h^2\varepsilon^2$$

Observing that $g(0) = 0$, we derive the conclusion of the Lemma: $\forall\, x_h \in \Pi_K(x + hf(x))$,

$$d\left(f(x), \frac{\Pi_K(x + hf(x)) - x}{h}\right) \;\leq\; \frac{\|x + hf(x) - x_h\|}{h} \;=\; \frac{d(x + hf(x), K)}{h} \;\leq\; \varepsilon \;\;\square$$

1.3 Statement of the Viability Theorems

Nagumo was the first one to prove the viability theorem for ordinary differential equations in 1942. This theorem was apparently forgotten, for it was rediscovered many times during the next twenty years.

We shall prove it when the subset $K \subset \mathbf{R}^n$ is *locally compact*:

Theorem 1.3.1 (Nagumo) *Let us assume that*

$$\begin{cases} i) & K \text{ is locally compact} \\ ii) & f \text{ is continuous from } K \text{ to } \mathbf{R}^n \end{cases} \tag{1.8}$$

Then K is locally viable under f if and only if K is a viability domain of f in the sense that

$$\forall\, x \in K,\;\; f(x) \;\in\; T_K(x)$$

or, equivalently, in terms of normal cones,

$$\forall\, x \in K,\; \forall\, p \in N_K(x),\;\; \langle p, f(x)\rangle \;\leq\; 0$$

Since the contingent cone to an open subset is equal to the whole space, an open subset is a viability domain of any map. So, it is viable under any continuous map because any open subset of a finite dimensional vector space is locally compact. The Peano existence theorem is then a consequence of Theorem 1.3.1.

Theorem 1.3.2 (Peano) *Let Ω be an open subset of a finite dimensional vector space $\mathbf{R}^n$ and $f : \Omega \mapsto \mathbf{R}^n$ be a continuous map.*

Then, for every $x_0 \in \Omega$, there exists $T > 0$ such that differential equation (1.1) has a solution on the interval $[0, T]$ starting at x_0.

If $C \subset K \subset \mathbf{R}^n$ is a closed subset of a closed subset K of a finite dimensional vector space, then $K \backslash C$ is locally compact, because for any $x \in K \backslash C$, there exists $r > 0$ such that $K \cap B(x, r) \subset \mathbf{R}^n \backslash C$. On the other hand,

$$\forall\, x \in K \backslash C,\ \ T_{K\backslash C}(x) \ = \ T_K(x)$$

Therefore, Theorem 1.3.1 implies

Theorem 1.3.3 *Let $C \subset K \subset \mathbf{R}^n$ is a closed subset of a closed subset K of a finite dimensional vector space $\mathbf{R}^n$ and $f : K \backslash C \mapsto \mathbf{R}^n$ be a continuous map. Then $K \backslash C$ is locally viable under f if and only if*

$$\forall\, x \in K \backslash C,\ \ f(x) \ \in \ T_K(x)$$

or, equivalently, in terms of normal cones,

$$\forall\, x \in K \backslash C,\ \forall\, p \in N_K(x),\ \ \langle p, f(x) \rangle \ \le \ 0$$

The interesting case from the viability point of view is the one when the viability subset is *closed*. This is possible because any closed subset of a finite dimensional vector space is locally compact. However, in this case, we derive from Theorem 1.3.1 a more precise statement.

Theorem 1.3.4 (Viability) *Let us consider a* closed *subset K of a finite dimensional vector space $\mathbf{R}^n$ and a* continuous *map f from K to $\mathbf{R}^n$. Then K is locally viable under f if and only if*

$$\forall\, x \in K,\ \ f(x) \ \in \ T_K(x)$$

If such is the case, then for every initial state $x_0 \in K$, there exist a positive T and a viable *solution on $[0, T[$ to differential equation (1.1) starting at x_0 such that*

$$\begin{cases} \textit{either} & T = \infty \\ \quad \textit{or} & T < \infty \quad \textit{and} \ \ \limsup_{t \to T-} \|x(t)\| = \infty \end{cases} \tag{1.9}$$

Further adequate information — a priori estimates on the growth of f — allows us to exclude the case when $\limsup_{t\to T-}\|x(t)\| = \infty$.

This is the case for instance when f is bounded on K, and, in particular, when K is bounded.

More generally, we can take $T = \infty$ when f enjoys linear growth:

Theorem 1.3.5 *Let us consider a subset K of a finite dimensional vector space $\mathbf{R}^n$ and a map f from K to $\mathbf{R}^n$. We assume that the map f is* continuous *from K to $\mathbf{R}^n$, that it has* linear growth *in the sense that*

$$\exists\, c > 0 \quad \textit{such that } \forall\, x \in K, \;\; \|f(x)\| \;\leq\; c(\|x\| + 1)$$

If

$$\forall\, x \in K, \;\; f(x) \;\in\; T_K(x)$$

then K is globally viable under f: For every initial state $x_0 \in K$, there exists a viable solution on $[0, \infty]$ *to differential equation (1.1) starting at x_0 and satisfying*

$$\forall\, t \geq 0, \;\; \|x(t)\| \;\leq\; \|x_0\|e^{ct} + e^{ct} - 1$$

1.4 Proofs of the Viability Theorems

We shall begin by proving Theorem 1.3.1. The necessary condition follows from Lemma 1.2.4.

For proving the sufficient condition, we begin by constructing approximate solutions by modifying the classical the Euler method to take into account the viability constraints, we then deduce from available estimates that a subsequence of these solutions converges uniformly to a limit, and finally check that this limit is a viable solution to differential equation (1.1.)

1. — **Construction of Approximate Solutions**

Since K is locally compact, there exists $r > 0$ such that the ball $B_K(x_0, r) := K \cap (x_0 + rB)$ is *compact*. When C is a subset, we set

$$\|C\| \;:=\; \sup_{v\in C} \|v\|$$

and

$$K_0 \;:=\; K \cap B(x_0, r), \; C \;:=\; B(f(K_0), 1), \; T \;:=\; \frac{r}{\|C\|}$$

We observe that C *is bounded* since K_0 *is compact.*

Let us consider the balls $B(x, \theta(x, \varepsilon))$ defined in Lemma 1.2.12 with $\varepsilon := \frac{1}{m}$. The compact subset K_0 can be covered by q balls $B(x_i, \eta(x_i, \frac{1}{m}))$. Taking $\theta :=$

$\min_{i=1,\ldots,q} \eta(x_i, \frac{1}{m}) > 0$, J the smallest integer larger than or equal to $\frac{T}{\theta}$ and setting $h := \frac{T}{J} \leq \theta$, we infer that

$$\forall\, x \in K_0, \quad d\left(f(x), \frac{\Pi_K(x + hf(x)) - x}{h}\right) \;\leq\; \frac{1}{m}$$

Starting from x_0, instead of defining recursively the sequence of elements $y_{j+1} := y_j + hf(y_j)$ as in the classical Euler method, we define recursively a sequence of elements

$$x_{j+1} \;:=\; x_j + hu_j \;\in\; \Pi_K(x_j + hf(x_j))$$

where $f(x_j)$ is replaced by u_j:

$$u_j \;\in\; \frac{\Pi_K(x_j + hf(x_j)) - x_j}{h} \;\subset\; C \text{ satisfies } \|f(x_j) - u_j\| \;\leq\; \frac{1}{m}$$

for keeping the elements $x_j \in K$.

The elements x_j belong to K_0, since they belong to K and

$$\|x_j - x_0\| \;\leq\; \sum_{i=0}^{i=j-1} \|x_{i+1} - x_i\| \;\leq\; jh\|C\| \;\leq\; T\|C\| = r$$

whenever $j \leq J$. We interpolate the sequence of elements x_j at the nodes jh by the piecewise linear functions $x_m(t)$ defined on each interval $[jh, (j+1)h[$ by

$$\forall\, t \in [jh, (j+1)h[, \quad x_m(t) \;:=\; x_j + (t - jh)u_j$$

We observe that this sequence satisfies the following estimates

$$\left\{\begin{array}{ll} i) & \forall\, t \in [0,T], \quad x_m(t) \in\; B(K_0, \varepsilon_m) \\ \\ ii) & \forall\, t \in [0,T], \quad \|x_m'(t)\| \leq \|C\| \end{array}\right. \tag{1.10}$$

Let us fix $t \in [\tau_m^j, \tau_m^{j+1}[$. Since $\|x_m(t) - x_m(\tau_m^j)\| = h_j\|u_j\| \leq \|C\|/m$, and since (x_j, u_j) belongs to $B(\mathrm{Graph}(f), \frac{1}{m})$ by Lemma 1.2.12, we deduce that these functions are approximate solutions in the sense that

$$\left\{\begin{array}{ll} i) & \forall\, t \in [0,T], \quad x_m(t) \in B(K_0, \varepsilon_m) \\ \\ ii) & \forall\, t \in [0,T], \;\; (x_m(t), x_m'(t)) \in B(\mathrm{Graph}(f), \varepsilon_m) \end{array}\right. \tag{1.11}$$

where $\varepsilon_m := \frac{\|C\|+1}{m}$ converges to 0.

2. — **Convergence of the Approximate Solutions**

Estimates (1.10) imply that for all $t \in [0,T]$, the sequence $x_m(t)$ remains in the compact subset $B(K_0, 1)$ and that the sequence $x_m(\cdot)$ is *equicontinuous*, because the derivatives $x_m'(\cdot)$ are bounded. We then deduce from Ascoli's Theorem[3] that it remains in a compact subset of the Banach space $\mathcal{C}(0,T;\mathbf{R}^n)$, and thus, that a subsequence (again denoted) $x_m(\cdot)$ converges uniformly to some function $x(\cdot)$.

3. — **The Limit is a Solution**

Condition (1.11)i) implies that

$$\forall\, t \in [0,T], \;\; x(t) \in K_0$$

i.e., that $x(\cdot)$ is viable.

Property (1.11)ii) implies that for almost every $t \in [0,T]$, there exist u_m and v_m converging to 0 such that

$$x_m'(t) \;=\; f(x_m(t) - u_m) + v_m$$

We thus deduce that for almost $t \geq 0$, $x_m'(t)$ converges to $f(x(t))$. On the other hand,

$$x_m(t) - x_m(s) \;=\; \int_s^t x_m'(\tau) d\tau$$

implies that $x_m'(t)$ converges almost every where to $x'(t)$. We thus infer that $x(\cdot)$ is a solution to the differential equation. □

Proof of Theorem 1.3.4 — First, K is locally compact since it is closed and the dimension of $\mathbf{R}^n$ is finite.

[3] Let us recall that a subset $\mathcal{H}$ of continuous functions of $\mathcal{C}(0,T;\mathbf{R}^n)$ is *equicontinuous* if and only if

$$\forall\, t \in [0,T],\; \forall\, \varepsilon > 0,\; \exists\, \eta \;:=\; \eta(\mathcal{H}, t, \varepsilon) \mid \forall\, s \in [t-\eta, t+\eta],\; \sup_{x(\cdot)\in\mathcal{H}} \|x(t) - x(s)\| \leq \varepsilon$$

Locally Lipschitz functions with the same Lipschitz constant form an equicontinuous set of functions. In particular, a subset of differentiable functions satisfying

$$\sup_{t\in[0,T]} \|x'(t)\| \leq c < +\infty$$

is equicontinuous.

Ascoli's Theorem states that a subset $\mathcal{H}$ of functions is *relatively compact* in $\mathcal{C}(0,T;\mathbf{R}^n)$ if and only if it is equicontinuous and satisfies

$$\forall\, t \in [0,T],\; \mathcal{H}(t) \;:=\; \{x(t)\}_{x(\cdot)\in\mathcal{H}} \text{ is compact.}$$

Second, we claim that starting from any x_0, there exists a maximal solution. Indeed, denote by $\mathcal{S}_{[0,T[}(x_0)$ the set of solutions to the differential equation defined on $[0,T[$.

We introduce the set of pairs $\{(T,x(\cdot))\}_{T>0,\, x(\cdot)\in\mathcal{S}_{[0,T[}(x_0)}$ on which we consider the order relation $\prec$ defined by

$$(T,x(\cdot)) \prec (S,y(\cdot)) \text{ if and only if } T \ \leq\ S\ \ \&\ \ \forall\, t\in[0,T[,\ \ x(t)=y(t)$$

Since every totally ordered subset has obviously a majorant, Zorn's Lemma implies that any solution $y(\cdot)\in\mathcal{S}_{[0,S[}(x_0)$ defined on some interval $[0,S[$ can be extended to a solution $x(\cdot)\in\mathcal{S}_{[0,T[}(x_0)$ defined on a maximal interval $[0,T[$.

Third, we have to prove that if T is finite, we cannot have

$$c \ :=\ \limsup_{t\to T-}\|x(t)\| \ <\ +\infty$$

Indeed, if $c<+\infty$, there would exist a constant $\eta\in]0,T[$ such that

$$\forall\, t\in[T-\eta,T[,\quad \|x(t)\|\ \leq\ c+1$$

Since f is continuous images on the compact subset $K\cap(c+1)B$, we infer that there exists a constant ρ such that for all $s\in[T-\eta,T[$, $\|f(x(s))\|\leq\rho$.

Therefore, for all $\tau,\sigma\in[T-\eta,T[$, we obtain:

$$\|x(\tau)-x(\sigma)\|\ \leq\ \int_\sigma^\tau\|f(x(s))\|ds\ \leq\ \rho|\tau-\sigma|$$

Hence the Cauchy criterion implies that $x(t)$ has a limit when $t\to T-$. We denote by $x(T)$ this limit, which belongs to K because it is closed. Equalities

$$x(T_k)\ =\ x_0+\int_0^{T_k}f(x(\tau))d\tau$$

imply that by letting $k\to\infty$, we obtain:

$$x(T)\ =\ x_0+\int_0^{T}f(x(\tau))d\tau$$

This means that we can extend the solution up to T and even beyond, since Theorem 1.3.1 allows us to find a viable solution starting at $x(T)$ on some interval $[T,S]$ where $S>T$. Hence c cannot be finite. $\square$

Proof of Theorem 1.3.5 — Since the growth of f is linear,

$$\exists\, c\geq 0,\quad \text{such that }\ \forall\, x\in\mathbf{R}^n,\ \ \|f(x)\|\leq c(\|x\|+1)$$

Therefore, any solution to differential equation (1.1) satisfies the estimate:

$$\|x'(t)\| \;\leq\; c(\|x(t)\| + 1)$$

The function $t \to \|x(t)\|$ being locally Lipschitz, it is almost everywhere differentiable. Therefore, for any t where $x(t)$ is different from 0 and differentiable, we have

$$\frac{d}{dt}\|x(t)\| \;=\; \left\langle \frac{x(t)}{\|x(t)\|}, x'(t) \right\rangle \;\leq\; \|x'(t)\|$$

These two inequalities imply the estimates:

$$\|x(t)\| \;\leq\; (\|x_0\| + 1)e^{ct} \;\;\&\;\; \|x'(t)\| \;\leq\; c(\|x_0\| + 1)e^{ct} \tag{1.12}$$

Hence, for any $T > 0$, we infer that

$$\limsup_{t \to T-} \|x(t)\| \;<\; +\infty$$

Theorem 1.3.4 implies that we can extend the solution on the interval $[0, \infty[$. $\square$

1.5 The Solution Map

Definition 1.5.1 (Solution Map) *We denote by $\mathcal{S}_f(x_0)$ the set of solutions to differential equation (1.1) and call the set-valued map $\mathcal{S}_f : x \rightsquigarrow \mathcal{S}_f(x)$ the* solution map *of f (or of differential inclusion (1.1).)*

Theorem 1.5.2 *Let us consider a finite dimensional vector space $\mathbf{R}^n$ and a* continuous map *$f : \mathbf{R}^n \mapsto \mathbf{R}^n$ with linear growth. Then the graph of the restriction of $\mathcal{S}_f|_L$ to any compact subset L is compact in $\mathbf{R}^n \times \mathcal{C}(0, \infty; \mathbf{R}^n)$ where the space $\mathcal{C}(0, \infty; \mathbf{R}^n)$ is supplied with the compact convergence topology.*

Proof — We shall show that the graph of the restriction $\mathcal{S}_f|_L$ of the solution map $\mathcal{S}_f$ to a compact subset L is compact. Let us choose a sequence of elements $(x_{0_n}, x_n(\cdot))$ of the graph of the solution map $\mathcal{S}_f$. They satisfy:

$$x'_n(t) = f(x_n(t)) \;\;\&\;\; x_n(0) = x_{0_n} \in L$$

A subsequence (again denoted) x_{0_n} converges to some $x_0 \in L$ because L is compact. By Theorem 1.3.5,

$$\forall\, n \geq 0,\;\; \|x_n(t)\| \;\leq\; (\|x_{0_n}\| + 1)e^{ct} \;\;\&\;\; \|x'_n(t)\| \;\leq\; c(\|x_{0_n}\| + 1)e^{ct}$$

Thus, by Ascoli's Theorem, the sequence $x_n(\cdot)$ is relatively compact in $\mathcal{C}(0,\infty;\mathbf{R}^n)$. We deduce from this, that a subsequence (again denoted) $x_n(\cdot)$ converges to a continuous function $x(\cdot)$ uniformly on compact intervals. Therefore, passing to the limit in equalities

$$x_m(t) \;=\; x_{0_n} + \int_0^t f(x_m(\tau))d\tau$$

we deduce that $x(\cdot)$ is a solution to the differential equation starting at x_0. □

1.6 Uniqueness Criteria

Whenever there exists a unique solution to differential equation $x' = f(x)$ starting from any initial state x_0, then viability and invariance properties of a closed subset K are naturally equivalent. This is one of the motivations for providing uniqueness criteria.

Definition 1.6.1 *We shall say that a map $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is* monotone *if there exists there exists $\mu \in \mathbf{R}$ such that*

$$\langle f(x_1) - f(x_2), x_1 - x_2\rangle \;\leq\; -\mu\|x_1 - x_2\|^2 \tag{1.13}$$

The interesting case is obtained when $\mu > 0$. When f is Lipschitz with constant λ, then it is monotone with $\mu = -\lambda$.

Theorem 1.6.2 *Let us consider a subset K of a finite dimensional vector space $\mathbf{R}^n$ and a continuous and monotone map f from $\mathbf{R}^n$ to $\mathbf{R}^n$. The solution to differential equation $x' = f(x)$ starting from x_0 is unique. If $x_i(\cdot)$ are two solutions to the differential equation $x' = f(x)$,then*

$$\|x_1(t) - x_2(t)\| \;\leq\; e^{-\mu t}\|x_1(0) - x_2(0)\|$$

Proof — Indeed, integrating the two sides of inequality

$$\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 \;=\; 2\langle f(x_1(t)) - f(x_2(t)), x_1(t) - x_2(t)\rangle \;\leq\; -2\mu\|x_1(t) - x_2(t)\|^2$$

yields

$$\|x_1(t) - x_2(t)\|^2 \;\leq\; e^{-2\mu t}\|x_1(0) - x_2(0)\|^2$$

1.7 Backward Viability

Definition 1.7.1 *The subset K is* locally backward viable *under f if for any $x \in K$, for any $t > 0$, there exist $s \in [0, t[$ and a solution $x(\cdot)$ to differential equation (1.1) such that*

$$\forall\, \tau \in [s,t],\ x(\tau) \in K \ \&\ x(t) = x$$

It is (globally) backward viable if we can take $s = 0$ in the above statement, and locally (resp. globally) backward invariant if for any $x \in K$, for any $t > 0$, for all solutions $x(\cdot)$ to differential equation (1.1), there exist $s \in [0, t[$ such that

$$\forall\, \tau \in [s,t],\ x(\tau) \in K \ \&\ x(t) = x$$

We now compare the invariance of a subset and the backward invariance of its complement:

Lemma 1.7.2 *A subset K is invariant under a map f if and only if its complement $K^c := \mathbf{R}^n \backslash K$ is backward invariant under f.*

Proof — To say that K is not invariant under f amounts to saying that there exist a solution $x(\cdot)$ to differential equation (1.1) and $T > 0$ such that

$$x(0) \ \in \ K \ \&\ x(T) \ \in \ \mathbf{R}^n \backslash K$$

and to say that $\mathbf{R}^n \backslash K$ is not backward invariant amounts to saying that there exist a solution $y(\cdot)$ to differential equation (1.1), $T > 0$ and $S \in [0, T[$ such that

$$y(S) \ \in \ K \ \&\ y(T) \ \in \ \mathbf{R}^n \backslash K$$

It is obvious that the first statement implies the second one by taking $y(\cdot) = x(\cdot)$ and $S = 0$. Conversely, the second statement implies the first one by taking $x(t) := y(t + S)$ and replacing T by $T - S > 0$ since $x(0) = y(S)$ belongs to K and $x(T - S) = y(T)$ belongs to $\mathbf{R}^n \backslash K$. □

It is also useful to relate backward viability and invariance under f to viability and invariance under $-f$:

Lemma 1.7.3 *Let us assume that f is continuous with linear growth. Then K is locally backward viable (resp. invariant) under f if and only if f is locally viable (resp. invariant) under $-f$.*

Proof — Let us check this statement for local viability. Assume that K is locally backward viable and infer that K is locally invariant under $-f$. Indeed, let $x \in K$. Then, for any $T > 0$, there exists $S \in [0, T[$ and a solution $x(\cdot)$ to differential

equation (1.1) viable in K on the interval $[S,T]$ and satisfying $x(T)=x$. Let $y(\cdot)$ be a solution to the differential equation $y'=-f(y)$ starting at $y(0)=x(S)$. Then the function $z(\cdot)$ defined by

$$z(t) \;=\begin{cases} x(T-t) & \text{if} \;\; t\in[0,T-S] \\ y(t+T-S) & \text{if} \;\; t\geq T-S \end{cases}$$

is a solution to the differential equation $z'=-f(z)$ starting at $z(0)=x(T)=x$ and viable in K on the interval $[0,T-S]$.

Conversely, assume that K is locally viable under $-f$ and check that K is locally backward invariant. Let $x\in K$, $T>0$ and one solution $x(\cdot)$ to differential equation $x'=-f(x)$ viable in K on $[0,R]$ where $R>0$. Let be any solution $y(\cdot)$ to $y'(t)=f(y(t))$ starting at x and set

$$z(t) \;=\begin{cases} x(T-t) & \text{if} \;\; t\in[0,T] \\ y(t-T) & \text{if} \;\; t\geq T \end{cases}$$

Hence $z(\cdot)$ to differential equation (1.1) satisfying $x(T)=x\in K$ and viable in K on the interval $[S,T]$ where $S:=\max(T-R,0)$. □

1.8 Time-Dependent Differential Equations

Theorem 1.8.1 *Let us consider a subset K of a finite dimensional vector space $\mathbf{R}^n$ and a map f from $\mathbf{R}_+\times K$ to $\mathbf{R}^n$. We assume that the map f is* continuous *from $\mathbf{R}_+\times K$ to $\mathbf{R}^n$, that it has* uniform linear growth *in the sense that*

$$\exists\, c>0 \quad \textit{such that} \;\; \forall\, t\geq 0,\; x\in K,\;\; \|f(t,x)\| \;\leq\; c(\|x\|+1)$$

If

$$\forall\, t\geq 0, \forall\, x\in K,\;\; f(t,x) \;\in\; T_K(x)$$

then K is globally viable under f: for every initial state $x_0\in K$, there exists a viable solution *on $[0,\infty]$ to differential equation*

$$x'(t) \;:=\; f(t,x(t))$$

starting at x_0 and satisfying

$$\forall\, t\geq 0,\;\; \|x(t)\| \;\leq\; \|x_0\|e^{ct}+e^{ct}-1$$

Assume moreover that f is uniformly monotone *in the sense that*[4] *there exists $\mu\in\mathbf{R}$ such that*

$$\langle f(t,x_1)-f(t,x_2),x_1-x_2\rangle \;\leq\; -\mu\|x_1-x_2\|^2 \tag{1.14}$$

[4] The interesting case is obtained when $\mu>0$. When f is uniformly Lipschitz with constant λ, then it is uniformly monotone with $\mu=-\lambda$.

If x_1 and x_2 are two initial states, then the solutions $x_i(\cdot)$ starting from x_i, $(i = 1, 2)$, satisfy

$$\|x_1(t) - x_2(t)\| \;\leq\; e^{-\mu t}\|x_1(0) - x_2(0)\|$$

Proof — We deduce the first statement from the standard trick which amounts to observing that a solution $x(\cdot)$ to the time-dependent differential equation $x' = f(t, x)$ starting at time 0 from the initial state x_0 if and only if $(\tau(\cdot), x(\cdot))$ is a solution to the system of differential equations

$$\begin{cases} i) & \tau'(t) \;=\; 1 \\ ii) & x'(t) \;=\; f(\tau(t), x(t)) \end{cases}$$

starting at time 0 from $(0, x_0)$. The solution $x(\cdot)$ is viable in K under f if and only if $(\tau(\cdot), x(\cdot))$ is viable in $\mathbf{R}_+ \times K$. By the Nagumo Theorem, this is equivalent to require that $(1, f(t, x)) \in T_{\mathbf{R}_+ \times K}(t, x)$, i.e., that $f(t, x)$ belongs to $T_K(x)$.

The proof of the second statement is same than the one of Theorem 1.6.2. $\square$

Chapter 2

Viability Kernels and Capture Basins

Introduction

When a closed subset K is not viable under a dynamical economy, then two questions arise naturally:

1. find solutions starting from K which *remain viable in K as long as possible, hopefully, forever*,

2. and starting outside of K, find solutions which *return to K as soon as possible, hopefully, in finite time*

Studying these questions leads to the concepts of

1. *viability kernel* of a subset K under a dynamical system, as the set of elements of K from which starts a solution viable in K,

2. *capture basin* of C, which is the set of points of K from which a solution reaches C in finite time,

3. when $C \subset K$, *viable-capture basin*, which is the subset of points of K from which starts a solution reaching C before leaving K.

We shall provide characterizations of these concepts and derive their properties we shall use later for solving some Hamilton-Jacobi equations and boundary-value problems for systems of first-order partial differential equations. For instance, if K is backward invariant and a repeller, the capture basin of C is the unique closed subset D satisfying

$$\begin{cases} i) & C \subset D \subset K \\ ii) & \forall\, x \in D\backslash C, \;\; f(x) \in T_D(x) \\ iii) & \forall\, x \in D, \;\; -f(x) \in T_D(x) \end{cases}$$

or, equivalently, by duality, the "normal conditions"

$$\begin{cases} i) & C \subset D \subset K \\ ii) & \forall\, x \in D\backslash C, \; \forall\, p \in N_D(x), \;\; \langle p, f(x) \rangle = 0 \\ iii) & \forall\, x \in D, \; \forall\, p \in N_D(x), \;\; \langle p, f(x) \rangle \geq 0 \end{cases}$$

2.1 Reachable, Viability and Capture Tubes

Definition 2.1.1 *Let $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ be a map and $C \subset \mathbf{R}^n$ be a subset. The* reachable map *$\vartheta_f(\cdot, x)$ is defined by*

$$\forall\, x \in \mathbf{R}^n, \;\; \vartheta_f(t,x) := \{x(t)\}_{x(\cdot)\in\mathcal{S}_f(x)}$$

We associate with it the reachable tube *$t \rightsquigarrow \vartheta_f(t, C)$ defined by*

$$\vartheta_f(t,C) := \{x(t)\}_{x(\cdot)\in\mathcal{S}_f(C)}$$

We derive the following properties:

Proposition 2.1.2 *The reachable map $t \rightsquigarrow \vartheta_f(t,x)$ enjoys the semi-group property: $\forall t, s \geq 0, \;\; \vartheta_f(t+s, x) = \vartheta_f(t, \vartheta_f(s,x))$.*

Furthermore,

$$(\vartheta_f(t,\cdot))^{-1} := \vartheta_{-f}(t,\cdot)$$

Therefore, the subset $\vartheta_{-f}(t,C)$ is the subset of elements $x \in \mathbf{R}^n$ which reach the subset at the prescribed time t.

If f is continuous with linear growth and $K \subset \mathbf{R}^n$ is closed, the graph of the reachable map $t \rightsquigarrow \vartheta_f(t,K)$ is closed.

Proof — The semi-group property is obvious. Let us prove the second one: If $y \in \vartheta_f(t,x)$, there exists a solution $x(\cdot)$ to the differential equation $x' = f(x)$ starting at x such that $y = x(t)$. We set $y(s) := x(t-s)$ if $s \in [0,t]$ and we choose any solution $y(\cdot)$ to the differential equation $y' \in -f(y)$ starting at x at time t for $s \geq t$. Then such a function $y(\cdot)$ is a solution to the differential equation $y' \in -f(y)$ starting at y and satisfying $y(t) = x$. This shows that $x \in \vartheta_{-f}(t,y)$.

The last statement is a consequence of Theorem 1.3.5. $\square$

Definition 2.1.3 *Let $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ be a map and $C \subset \mathbf{R}^n$ be any subset.*

1. *The subset $\mathrm{Viab}_f(C,T)$ of initial states $x_0 \in C$ such that one solution $x(\cdot)$ to differential equation $x' = f(x)$ starting at x_0 is viable in C for all $t \in [0,T]$ is called the T-*viability kernel *and the subset $\mathrm{Viab}_f(C) := \mathrm{Viab}_f(C,\infty)$ is called the* viability kernel *of C under f. A subset C is a repeller if its viability kernel is empty.*

2. *The subset $\mathrm{Capt}_f(C,T)$ of initial states $x_0 \in \mathbf{R}^n$ such that C is reached before T by one solution $x(\cdot)$ to differential equation $x' = f(x)$ starting at x_0 is called the T-*capture basin *and*

$$\mathrm{Capt}_f(C) \;:=\; \bigcup_{T>0} \mathrm{Capt}_f(C,T)$$

is said to be the capture basin *of C.*

3. *When $C \subset K$, the* viable-capture basin *$\mathrm{Capt}_f^K(C)$ of C in K by f is the set of initial states $x_0 \in \mathbf{R}^n$ from which starts at least one solution to the $x' = f(x)$ viable in K until it reaches C in finite time.*

Remarks — We observe that if $T_1 \le T_2$,

$$\left\{ \begin{array}{ccccc} \mathrm{Viab}_f(C) & \subset & \mathrm{Viab}_f(C,T_2) & \subset & \mathrm{Viab}_f(C,T_1) \\ \cap & & \cap & & \cap \\ \mathrm{Capt}_f(C,0) & = & C & = & \mathrm{Viab}_f(C,0) \\ \cap & & \cap & & \cap \\ \mathrm{Capt}_f(C) & \supset & \mathrm{Capt}_f(C,T_2) & \supset & \mathrm{Capt}_f(C,T_1) \end{array} \right.$$

One can write

$$\mathrm{Capt}_f(C,T) \;=\; \bigcup_{t\in[0,T]} \vartheta_{-f}(t,C)$$

We point out the following obvious properties:

Lemma 2.1.4 *Let $C \subset K$ be closed subsets. The capture basin $\mathrm{Capt}_f(C)$ is smallest backward invariant containing C and $\mathrm{Capt}_f(C)\backslash C$ is locally viable. The capture basin of any union of subsets C_i $(i \in I)$ is the union of the capture basins of the C_i. When $C \subset K$ where K is assumed to be backward invariant, then the viable-capture basin satisfies*

$$\mathrm{Capt}_f^K(C) \;=\; \mathrm{Capt}_f(C) \;\subset\; K$$

The complement $\mathrm{Capt}_f^K(C)\backslash C$ of C in the viable-capture basin $\mathrm{Capt}_f^K(C)$ is locally viable.

Proof — Indeed, whenever K is backward invariant, each backward reachable set $\vartheta_{-f}(t,C)$ is contained in K, so that

$$\mathrm{Capt}_f(C) \;=\; \bigcup_{t\geq 0} \vartheta_{-f}(t,C) \;\subset\; K$$

Since the intersection of backward invariant subsets is backward invariant, the capture basin is contained in the smallest backward invariant subset containing C. The semi group property implies that the capture basin, which is the union of the backward reachable subsets, is backward invariant.

When $C \subset K$ where K is assumed to be backward invariant, then the capture basin

$$\bigcup_{t\geq 0} \vartheta_{-f}(t,C)$$

is contained in K.

If x belongs to $\mathrm{Capt}_f^K(C)\backslash C$, then there exists a solution to the differential equation $x' = f(x)$ starting from x which reaches C before leaving K, and thus, which is viable in $\mathrm{Capt}_f^K(C)\backslash C$ on some nonempty interval. □

Proposition 2.1.5 *The viability kernel* $\mathrm{Viab}_f(C)$ *of* C *under* f *is the largest subset of* C *viable under* f.

Furthermore, $C\backslash\mathrm{Viab}_f(C)$ *is a repeller and* $\mathrm{Viab}_f(C)\backslash\partial C$ *is locally backward invariant.*

Proof — Every subset $L \subset C$ viable under f is obviously contained in the viability kernel $\mathrm{Viab}_f(C)$ of C under f.

On the other hand, if $x(\cdot)$ is a solution to the differential equation $x' = f(x)$ viable in C, then for all $t > 0$, the function $y(\cdot)$ defined by $y(\tau) := x(t+\tau)$ is also a solution to the differential equation, starting at $x(t)$, viable in C.

Therefore, for any element $x_0 \in \mathrm{Viab}_f(C)$, there exists a viable solution $x(\cdot)$ to the differential equation starting from x_0, and thus, for all $t \geq 0$, $x(t) \in \mathrm{Viab}_f(C)$, so that it is viable under f.

Let us assume that $\mathrm{Viab}_f(C)\backslash\partial C$ is not locally backward invariant: There would exist $x \in \mathrm{Viab}_f(C)\backslash\partial C$, $T > 0$ and a solution $x(\cdot)$ to the differential equation $x' = f(x)$ satisfying $x(T) = x$ such that for all $S < T$, there exist $S' \in [S,T]$ such that $x(S')$ belong to the union of ∂C and of the complement of the viability kernel $\mathrm{Viab}_f(C)$. Since $x(T) = x$ does not belong to the boundary ∂C of C, we know that for S close enough to T, $x([S,T]) \cap \partial C = \emptyset$. Hence $x(S')$ does not belong to the boundary of C, so that it belongs to complement of the viability kernel $\mathrm{Viab}_f(C)$, and thus, the solution $x(\cdot)$ starting from $x(S') \in \mathrm{Viab}_f(C)$ at time S' should leave C in finite time, a contradiction. □

2.2 Hitting and Exit Times

Definition 2.2.1 *We say that the* hitting functional *(or* minimal time functional*) associating with $x(\cdot)$ its* hitting time $\omega_C(x(\cdot))$ *is defined by*

$$\omega_C(x(\cdot)) \ := \ \inf\{t \in [0,+\infty[\ | \ x(t) \in C\}$$

and the function $\omega_C^{f^\flat} : C \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\omega_C^{f^\flat}(x) \ := \ \inf_{x(\cdot)\in\mathcal{S}_f(x)} \omega_C(x(\cdot))$$

is called the (lower) hitting function *or* minimal time function*. In the same way, the* exit functional *is defined by*

$$\tau_C(x(\cdot)) \ := \ \omega_{\mathbf{R}^n\backslash C}(x(\cdot)) \ := \ \inf\{t \in [0,+\infty[\ | \ x(t) \notin C\}$$

and the function $\tau_C^{f^\sharp} : C \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\tau_C^{f^\sharp}(x) \ := \ \sup_{x(\cdot)\in\mathcal{S}_f(x)} \tau_C(x(\cdot))$$

is called the (upper) exit function*.*

Let $C \subset K \subset \mathbf{R}^n$ be two closed subsets. We also introduce the function

$$\gamma_{(K,C)}^{f^\flat}(x) \ := \ \inf_{x(\cdot)\in\mathcal{S}_f(x)} (\omega_C(x(\cdot)) - \tau_K(x(\cdot)))$$

We observe that if $C = K$, $\omega_C(x(\cdot)) = 0$ for all solutions $x(\cdot) \in \mathcal{S}_f(x)$ starting from K and thus, that

$$\gamma_{(K,K)}^{f^\flat}(x) \ = \ -\tau_K^{f^\sharp}(x)$$

To say that $K\backslash C$ is a repeller amounts to saying that for every solution $x(\cdot) \in \mathcal{S}_f(x)$ starting from $x \in K\backslash C$, $\min(\omega_C(x(\cdot), \tau_K(x(\cdot)))) < +\infty$ and to say that K is a repeller under f amounts to saying that the exit function $\tau_K^{f^\sharp}$ is finite on K.

Lemma 2.2.2 *When K is closed, the exit functional τ_K is upper semicontinuous and the hitting functional ω_K is lower semicontinuous when $\mathcal{C}(0,\infty;\mathbf{R}^n)$ is supplied with the compact convergence topology.*

Proof — For proving that the exit functional τ_K is upper semicontinuous, we shall check that the subsets $\{x(\cdot) \mid \tau_K(x(\cdot)) < T\}$ are open for the pointwise convergence, and thus, the compact convergence. Let $x_0(\cdot)$ belong to such a set when it is not empty. Since $x_0(T)$ does not belong K which is closed, there exist

$\alpha > 0$ such that $B(x_0(T), \alpha) \cap K = \emptyset$. Then the set of continuous functions $x(\cdot)$ such that $x(T) \in \overset{\circ}{B} (x_0(T), \alpha)$ is open and satisfy $\tau_K(x(\cdot)) < T$.

For proving that the hitting functional is lower semicontinuous, we shall check that the subsets $\{x(\cdot) \mid \omega_K(x(\cdot)) \leq T\}$ are closed. Let $x_n(\cdot)$ satisfying $\omega_K(x_n(\cdot)) \leq T$ converge to $x(\cdot)$ uniformly on compact intervals. For any $\varepsilon > 0$, one can find $t_n \leq T + \varepsilon$ such that $x_n(t_n)$ belongs to K. A subsequence (again denoted by) t_n converges to some t. Since $x_n(\cdot)$ converges uniformly to $x(\cdot)$ on $[t - \varepsilon, T + \varepsilon]$, we deduce that $x(t)$ is the limit of $x_n(t_n) \in K$ and thus, that $x(t)$ belongs to the closed subset K and thus, that $\omega_K(x(\cdot)) \leq T + \varepsilon$. Letting ε converge to 0, we infer that $\omega_K(x(\cdot)) \leq T$. $\square$

We deduce the following properties of these hitting and exit functions:

Proposition 2.2.3 *Let $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ be a continuous map with linear growth and $C \subset K \subset \mathbf{R}^n$ be two closed subsets. Assume that K is a repeller.*

The function $\gamma^{f^\flat}_{(K,C)}$ is lower semicontinuous and for any $x \in K$, there exists a solution $x_{(K,C)}(\cdot) \in \mathcal{S}_f(x)$ satisfying

$$\gamma^{f^\flat}_{(K,C)}(x) \;:=\; \omega_C(x_{(K,C)}(\cdot)) - \tau_K(x_{(K,C)}(\cdot)))$$

In the same way, the hitting function $\omega^{f^\flat}_K$ is lower semicontinuous and the exit function $\tau^{f^\sharp}_K$ is upper semicontinuous. Furthermore, for any $x \in \mathrm{Dom}(\omega^{f^\flat}_K)$, there exists one solution $x^\flat(\cdot) \in \mathcal{S}_f(x)$ which hits K as soon as possible

$$\omega^{f^\flat}_K(x) \;=\; \omega_K(x^\flat(\cdot))$$

and for any $x \in \mathrm{Dom}(\tau^{f^\sharp}_K)$, there exists one solution $x^\sharp(\cdot) \in \mathcal{S}_f(x)$ which remains viable in K as long as possible:

$$\tau^{f^\sharp}_K(x) \;=\; \tau_K(x^\sharp(\cdot))$$

Proof — Since the function $x(\cdot) \mapsto \omega_C(x(\cdot)) - \tau_K(x(\cdot))$ is lower semicontinuous on $\mathcal{C}(0, \infty, \mathbf{R}^n)$ supplied with the compact convergence by Theorem 2.2.2, we deduce first from Theorem 1.5.2 that the infimum is reached by a solution $x_{(K,C)}(\cdot) \in \mathcal{S}_f(x)$ because the set $\mathcal{S}_f(x)$ is compact and second, that this function $\gamma^{f^\flat}_{(K,C)}$ is lower semicontinuous, by checking that the subsets $\{x \in K \mid \gamma^{f^\flat}_{(K,C)}(x) \leq T\}$ are closed. Indeed, let us consider a sequence of elements x_n of such a subset converging to x. There exist solutions $x_n(\cdot) \in \mathcal{S}_f(x)$ such that

$$\gamma_{(K,C)}(x_n(\cdot)) \;\leq\; \gamma^{f^\flat}_{(K,C)}(x_n) + \frac{1}{n} \;\leq\; T + \frac{1}{n}$$

On the other hand, since x_n belongs to the compact ball $B(x,1)$, Theorem 1.5.2 implies that a subsequence (again denoted by) $x_n(\cdot)$ converges to some solution $x(\cdot) \in \mathcal{S}_f(x)$ uniformly on compact intervals. Since the functional $\gamma_{(K,C)}$ is lower semicontinuous, we infer that

$$\gamma^{f^\flat}_{(K,C)}(x) \;\le\; \gamma_{(K,C)}(x(\cdot)) \;\le\; \liminf_{n\to+\infty} \gamma_{(K,C)}(x_n(\cdot)) \;\le\; T$$

In particular, taking $C := K$, we observe that $\gamma^{f^\flat}_{(K,K)} = -\tau^{f^\sharp}_K$, and thus, we deduce the upper semicontinuity of the exit function. The same proof shows that the hitting function $\omega^{f^\flat}_C$ is lower semicontinuous. $\square$

Viability kernels and capture basins can be characterized in terms of exit and hitting functionals:

Theorem 2.2.4 *If $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is continuous with linear growth and $C \subset K \subset \mathbf{R}^n$ are closed subsets, then*

$$\begin{cases} \mathrm{Capt}_f(K,T) \;=\; \left\{x \in \mathbf{R}^n \mid \omega^{f^\flat}_K(x) \;\le\; T\right\} \;\&\; \mathrm{Capt}_f(K,T) \;=\; \mathrm{Dom}(\omega^{f^\flat}_K) \\ \mathrm{Capt}^K_f(C,T) \;=\; \left\{x \in \mathbf{R}^n \mid \gamma^{f^\flat}_{(K,C)}(x) \;\le\; 0\right\} \\ \mathrm{Viab}_f(K,T) \;:=\; \left\{x \in \mathbf{R}^n \mid \tau^\sharp_K(x) \;\ge\; T\right\} \end{cases}$$

In particular, the T-viability kernels $\mathrm{Viab}_f(K,T)$ of a closed subset $K \subset \mathbf{R}^n$, the T-capture basins of K under f and the viable-capture basin $\mathrm{Capt}^K_f(C,T)$ are closed.

Proof — The subset of initial states $x \in \mathbf{R}^n$ such that K is reached before T by a solution $x(\cdot)$ to the differential equation $x' = f(x)$ starting at x is obviously contained in the subset $\left\{x \in \mathbf{R}^n \mid \omega^{f^\flat}_K(x) \;\le\; T\right\}$.

Conversely, consider an element x satisfying $\omega^{f^\flat}_K(x) \;\le\; T$. Hence the solution $x^\flat(\cdot) \in \mathcal{S}_f(x)$ such that $\omega_K(x^\flat) = \omega^{f^\flat}_K(x) \le T$ belongs to the T-capture basin.

Now, to say that a solution $x(\cdot) \in \mathcal{S}_f(x)$ is viable in K until it reaches the target C means that $\omega_C(x(\cdot)) \le \tau_K(x(\cdot))$. Therefore, x belongs to the viable-capture basin $\mathrm{Capt}^K_f(C)$ if and only if $\gamma^{f^\flat}_{(K,C)}(x) \le 0$.

The proof of the characterization of the T-viability kernel $\mathrm{Viab}_f(K,T)$ as upper sections of the exit time function $\tau^\sharp_K$ is analogous.

The topological properties then follow from the semicontinuity properties of the above functions stated in Proposition 2.2.3. $\square$

2.3 Characterization of the Viability Kernel

We deduce at once the following consequence:

Theorem 2.3.1 *Let $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ be a continuous map with linear growth and $K \subset \mathbf{R}^n$ be a closed subset. Then the viability kernel is the largest closed subset $D \subset K$ viable under f, or, equivalently, the largest closed subset of K satisfying*

$$\begin{cases} i) & D \subset K \\ ii) & \forall\, x \in D, \;\; f(x) \in T_D(x) \end{cases} \tag{2.1}$$

or, equivalently, in terms of normal cones, the largest closed subset of K satisfying

$$\begin{cases} i) & D \subset K \\ ii) & \forall\, x \in D, \; \forall\, p \in N_D(x) \; \langle p, f(x)\rangle \;\leq\; 0 \end{cases} \tag{2.2}$$

Furthermore, the viability kernel satisfies the following properties

$$\forall\, x \in \mathrm{Viab}_f(K)\backslash\partial K, \;\; f(x) \;\in\; T_{\mathrm{Viab}_{f(K)}}(x) \cap -T_{\mathrm{Viab}_{f(K)}}(x)$$

or, equivalently, in terms of normal cones,

$$\forall\, x \in \mathrm{Viab}_f(K)\backslash\partial K, \; \forall\, p \in N_{\mathrm{Viab}_{f(K)}}(x), \;\; \langle p, f(x)\rangle \;=\; 0$$

Proof — The first property follows from the Nagumo Theorem characterizing viable subsets in terms of tangential and/or normal conditions. The second property translates the fact that $\mathrm{Viab}_f(K)\backslash\partial K$ is locally backward invariant, and thus, locally backward viable. □

Proposition 2.3.2 *Let $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ be a continuous map with linear growth and $K \subset \mathbf{R}^n$ be a closed subset. If $M \subset \mathbf{R}^n\backslash\mathrm{Viab}_f(K)$ is compact, then, for every $x \in M$ and every solution $x(\cdot) \in \mathcal{S}_f(x)$, there exists $t \in [0, \sup_{x\in M} \tau_K^{f^\sharp}(x)]$ such that $x(t) \notin K$.*

Proof — Indeed, M being compact and the exit function being upper semicontinuous, then $\sup_{x\in M} \tau_K^{f^\sharp}(x)$ is finite because, for each $x \in M$, $\tau_K^{f^\sharp}(x)$ is finite. □

In particular,

Proposition 2.3.3 *Let us assume that K is a compact and that $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is continuous with linear growth. Then either the viability kernel of K is not empty or K is a repeller, and in this case, $\overline{T} := \sup_{x\in K} \tau_K^{f^\sharp}(x)$ is finite and satisfies*

$$\mathrm{Viab}_f(K, \overline{T}) \;\neq\; \emptyset \;\;\&\;\; \forall\, T > \overline{T}, \;\; \mathrm{Viab}_f(K, T) \;=\; \emptyset$$

Proof — When K is a repeller, the exit function is finite. Being compact, $\overline{T} := \sup_{x\in K} \tau_K^{f^\sharp}(x)$ is thus finite and achieves its maximum at some $\bar{x}$. By Theorem 2.2.3, there exists a solution $\bar{x}(\cdot) \in \mathcal{S}_f(\bar{x})$ such that $\tau_K(\bar{x}(\cdot)) = \tau_K^{f^\sharp}(\bar{x}) = \overline{T}$. □

In other words, when K is a compact repeller, there exists a smallest nonempty T-viability kernel of K, the "viability core", so to speak, because it is the subset of initial states from which one solution which enjoys the longest "life expectation" $\overline{T}$ in K. The viability kernel, when it is nonempty, is the viability core with infinite life expectation.

2.4 Characterization of Viable-Capture Basins

Let $C \subset K$ be a closed subset of a closed subset $K \subset \mathbf{R}^n$.

Theorem 2.4.1 *Let us assume that f is continuous with linear growth and that K is a closed repeller under f. Then the viable-capture basin $\mathrm{Capt}_f^K(C)$ is the largest closed subset D satisfying*

$$\begin{cases} i) & C \subset D \subset K \\ ii) & D\backslash C \text{ is locally viable under } f \end{cases}$$

or, equivalently, is the largest closed subset D satisfying

$$\begin{cases} i) & C \subset D \subset K \\ ii) & \forall\, x \in D\backslash C, \quad f(x) \in T_D(x) \end{cases} \tag{2.3}$$

or, equivalently, in terms of normal cones, is the largest closed subset D satisfying

$$\begin{cases} i) & C \subset D \subset K \\ ii) & \forall\, x \in D\backslash C,\ \forall\, p \in N_D(x), \quad \langle p, f(x)\rangle \le 0 \end{cases} \tag{2.4}$$

Proof of Theorem 2.4.1 — Assume that a closed subset D such that $C \subset D \subset K$ is a repeller under f such that $D\backslash C$ is locally viable under f and let us check that it is contained in $\mathrm{Capt}_f^K(C)$. Since $C \subset \mathrm{Capt}_f^K(C)$, let x belong to $D\backslash C$ and show that it belongs to $\mathrm{Capt}_f^K(C)$. Since K is a repeller, all solutions starting from x leave $D\backslash C$ in finite time. At least one of them, the solution $x^\sharp(\cdot) \in \mathcal{S}_f(x)$ which maximizes $\tau_D(x(\cdot))$:

$$\tau_D^{f^\sharp}(x) := \sup_{x(\cdot)\in\mathcal{S}_f(x)} \tau_D^{f^\sharp}(x(\cdot)) = \tau_D^{f^\sharp}(x^\sharp(x))$$

leaves $D\backslash C$ through C. This solution exists by Proposition 2.2.3 since D is closed and f is continuous with linear growth. Then we claim that $x^\sharp := x^\sharp(\tau_D^{f^\sharp}(x))$ belongs to C. If not, $D\backslash C$ being locally viable, one could associate with $x^\sharp \in D\backslash C$ a solution $y(\cdot) \in \mathcal{S}_f(x^\sharp)$ and $T > 0$ such that $y(\tau) \in D\backslash C$ for all $\tau \in [0,T]$. Concatenating this solution to $x^\sharp(\cdot)$, we obtain a solution viable in D on an interval $[0, \tau_D^{f^\sharp}(x) + T]$, which contradicts the definition of $x^\sharp(\cdot)$. Furthermore, $x^\sharp(\cdot)$ is viable in K since $D \subset K$. This implies that $D \subset \mathrm{Capt}_f^K(C)$.

The viable-capture basin being a closed subset such that $\mathrm{Capt}_f^K(C)\backslash C$ is locally viable, we conclude that it is the largest closed subset D of K containing C such that $D\backslash C$ is locally viable under f.

Since f is continuous and since $D\backslash C$ is locally compact, the Viability Theorem 1.3.3 states that $D\backslash C$ is locally viable if and only if (2.3)ii) or (2.4)ii) holds true. □

2.5 Characterization of Capture Basins

Theorem 2.5.1 *Let us assume that the closed subset $K \subset \mathbf{R}^n$ is a repeller under f and backward invariant, that $C \subset K$ is closed and that f is continuous with linear growth.*

Then the capture basin $\mathrm{Capt}_f(C)$ *is the* unique *closed subset D which satisfies*

$$\begin{cases} i) & C \subset D \subset K \\ ii) & D\backslash C \text{ is locally viable under } f \\ iii) & D \text{ is backward invariant under } f \end{cases}$$

If we assume furthermore that f is Lipschitz, it is the unique closed subset satisfying the "tangential conditions"

$$\begin{cases} i) & C \subset D \subset K \\ ii) & \forall\, x \in D\backslash C, \ \ f(x) \in T_D(x) \\ iii) & \forall\, x \in D, \ \ -f(x) \in T_D(x) \end{cases}$$

or, equivalently, by duality, the "normal conditions"

$$\begin{cases} i) & C \subset D \subset K \\ ii) & \forall\, x \in D\backslash C,\ \forall\, p \in N_D(x), \ \ \langle p, f(x)\rangle = 0 \\ iii) & \forall\, x \in D,\ \forall\, p \in N_D(x), \ \ \langle p, f(x)\rangle \geq 0 \end{cases}$$

Proof — Since $C \subset K$ and K is backward invariant, we already observed that

$$\mathrm{Capt}_f^K(C) = \mathrm{Capt}_f(C) \subset K$$

Since K is a repeller, the capture basin $\mathrm{Capt}_f(C)$ is closed: For that purpose, let $x_n \in \mathrm{Capt}_f(C)$ converge to x and infer that x belongs to the capture basin of C. There exist t_n and $c_n := \vartheta_f(t_n, x_n)$ which belongs to C. Since K is a repeller, the exit function $\tau_K^{f^\sharp}$ is finite, and, being upper semicontinuous thanks to Lemma 2.2.2, $T := \sup_{x_n \in B(x,1)} \tau_K^{f^\sharp}(x_n) < +\infty$. Since $C \subset K$, we infer that $t_n \le T < +\infty$. Hence we can extract a subsequence (again denoted by) t_n converging to some $t \in [0, T]$, so that, the reachable map being continuous, c_n converges to $c = \vartheta_f(t, x)$ which belongs to C. Hence x belongs to $\mathrm{Capt}_f(C)$, which is then closed.

Therefore, the capture basin $\mathrm{Capt}_f(C)$ is a closed subset, backward invariant and locally viable.

If D is closed and backward invariant, we infer that $\mathrm{Capt}_f(C) \subset D$. By Theorem 2.4.1, if $D \backslash C$ is locally viable, we know that

$$\mathrm{Capt}_f(C) \ = \ \mathrm{Capt}_f^K(C) \ \subset \ D$$

Hence, the capture basin $\mathrm{Capt}_f(C)$ is the unique closed subset, backward invariant and locally viable between C and K. $\square$

2.6 Stability Properties

Let us consider now a sequence of closed subsets K_n viable under a map f. *Is the upper limit of these closed subsets still viable under f?* The answer is positive.

Theorem 2.6.1 *Let us assume that $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is continuous with linear growth. Then the upper limit of a sequence of subsets viable under f is still viable under f and the upper limit of the viability kernels of K_n is contained in the viability kernel of the upper limit:*

$$\mathrm{Limsup}_{n\to+\infty}\mathrm{Viab}_f(C_n) \ \subset \ \mathrm{Viab}_f(\mathrm{Limsup}_{n\to+\infty}C_n) \tag{2.5}$$

In particular, the intersection of a decreasing family of closed viability domains is a closed viability domain.

Proof — We shall prove that the upper limit $K^\sharp$ of a sequence of subsets K_n viable under f is still viable under f.

Let x belong to $K^\sharp$. It is the limit of a subsequence $x_{n'} \in K_{n'}$. Since the subsets K_n are viable under f, there exist solutions $y_{n'}(\cdot)$ to differential equation $x' = f(x)$ starting at $x_{n'}$ and viable in $K_{n'}$. Theorem 1.5.2 implies that a subsequence (again denoted) $y_{n'}(\cdot)$ converges uniformly on compact intervals to a solution $y(\cdot)$ to differential equation $x' = f(x)$ starting at x. Since $y_{n'}(t)$ belongs to $K_{n'}$ for all n', we deduce that $y(t)$ does belong to $K^\sharp$ for all $t > 0$. $\square$

Theorem 2.6.2 *Let us consider a sequence of closed subsets C_n.*

1. *If the map f is continuous with linear growth and if the subsets C_n are contained in a closed repeller K, then*

$$\text{Limsup}_{n\to+\infty}\text{Capt}_f^K(C_n) \;\subset\; \text{Capt}_f^K(\text{Limsup}_{n\to+\infty}C_n) \tag{2.6}$$

2. *If the map f is furthermore monotone and K is backward invariant, then*

$$\text{Capt}_f(\text{Liminf}_{n\to+\infty}C_n) \;\subset\; \text{Liminf}_{n\to+\infty}\text{Capt}_f(C_n) \tag{2.7}$$

Proof — For proving inclusion (2.6), we consider the limit $x := \lim_{n\to+\infty} x_n$ of elements x_n of $\text{Capt}_f^K(C_n)$. Let us consider solutions $x_n(\cdot)$ satisfying

$$t_n \;:=\; \omega_{C_n}(x_n(\cdot)) \;\leq\; \tau_K(x_n(\cdot)) \;\leq\; T \;:=\; \sup_{y\in B(x,1)\cap K} \tau_K^{f^\sharp}(y)$$

where T is finite because K is a repeller.

Therefore, a subsequence (again denoted by) t_n converges to some $t \leq T$ and another subsequence (again denoted by) $x_n(\cdot)$ converges uniformly on compact intervals to some solution $x(\cdot)$ starting from x. Since $x_n(t_n)$ belongs to C_n and converges to $x(t)$, we infer that $x(t)$ belongs to the upper limit $C^\sharp$ of the C_n. Hence

$$\omega_{C^\sharp}(x) \;\leq\; \lim_{n\to+\infty} t_n \;\leq\; \limsup_{n\to+\infty} \tau_K(x_n(\cdot)) \;\leq\; \tau_K(x(\cdot))$$

and thus, x belongs to the viable-capture basin of $C^\sharp$.

We now prove (2.7). Let $C^\flat$ denote the lower limit of the subsets C_n and let us consider x an element of $\text{Capt}_f(C^\flat)$, a solution $x(\cdot)$ starting from x and reaching $C^\flat$ at time T at $c := x(T)$. Hence $y(t) := x(T-t)$ is a solution to the backward differential equation $y' = -f(y)$ starting at c and satisfying $y(T) = x$. Since $c = \lim_{n\to+\infty} c_n$ where $c_n \in C_n$, Theorem 1.6.2 states that the solutions $y_n(\cdot)$ to the differential equation $y' = -f(y)$ starting from c_n satisfy

$$\|y(t) - y_n(t)\| \;\leq\; e^{-\mu t}\|c - c_n\|$$

Then $x_n := y_n(T) \in \text{Capt}_f(C_n)$ converges to $x = y(T) \in \text{Capt}_f(C^\flat)$. $\square$

Chapter 3

Epiderivatives and Subdifferentials

For reasons motivated by optimization theory, Lyapunov stability, control theory, Hamilton-Jacobi equations and mathematical morphology, the order relation on $\mathbf{R}$ is involved. This leads us to associate with an extended functions $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ its epigraph instead of its graph. It actually happens that the properties of the extended functions $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ are actually properties of their *epigraphs*. This "epigraphical point of view" is the key to "variational analysis" and to our treatment of Hamilton-Jacobi inequalities.

In particular, one can define the following concepts:

1. The epigraph of the *lower epilimit* of a sequence of extended functions $\mathbf{u}_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ is the upper limit of the epigraphs of the f_n,

2. *The contingent epiderivative* $D_\uparrow \mathbf{u}(x)$ at x is the lower epigraphical limit of the difference quotients $\nabla_h \mathbf{u}(x)$, so that the epigraph of the contingent epiderivative is the contingent cone to the epigraph of $\mathbf{u}$.

By duality, the normal cones to the epigraph of an extended function yields the concept of **subdifferential**, which is in particular used in the concepts of viscosity solutions to Hamilton-Jacobi equations.

3.1 Extended Functions and their Epigraphs

A function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ is called an *extended (real-valued) function*. Its *domain* is the set of points at which $\mathbf{v}$ is finite:

$$\mathrm{Dom}(\mathbf{v}) \ := \ \{x \in X \ \mid \ \mathbf{v}(x) < +\infty\}$$

A function is said to be *nontrivial* if its domain is not empty. Any function $\mathbf{v}$ defined on a subset $K \subset X$ can be regarded as the extended function $\mathbf{v}_K$ equal to $\mathbf{v}$ on K and to $+\infty$ outside of K, whose domain is K.

Since the order relation on the real numbers is involved in the definition of the Lyapunov property (as well as in minimization problems and other dynamical inequalities), we no longer characterize a real-valued function by its graph, but rather by its *epigraph* indexepigraph

$$\mathcal{E}p(\mathbf{v}) \;:=\; \{(x,\lambda) \in X \times \mathbf{R} \mid \mathbf{v}(x) \le \lambda\}$$

The *hypograph* of a function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{-\infty\}$ is defined in a symmetric way by

$$\mathcal{H}yp(\mathbf{v}) \;:=\; \{(x,\lambda) \in X \times \mathbf{R} \mid \mathbf{v}(x) \ge \lambda\} \;=\; -\mathcal{E}p(-\mathbf{v})$$

The graph of a real-valued (finite) function is then the intersection of its epigraph and its hypograph.

We also remark that some properties of a function are actually properties of their epigraphs. For instance, *an extended function* $\mathbf{v}$ *is convex (resp. positively homogeneous) if and only if its epigraph is convex (resp. a cone).* The epigraph of $\mathbf{v}$ is closed if and only if $\mathbf{v}$ is lower semicontinuous:

$$\forall\, x \in X, \;\; \mathbf{v}(x) \;=\; \liminf_{y \to x} \mathbf{v}(y)$$

We recall the convention $\inf(\emptyset) := +\infty$.

Lemma 3.1.1 *Consider a function* $\mathbf{v} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$*. Its epigraph is closed if and only if*

$$\forall\, x \;\in\; X, \;\; \mathbf{v}(x) \;=\; \liminf_{x' \to x} \mathbf{v}(x')$$

Assume that the epigraph of $\mathbf{v}$ *is a closed cone. Then the following conditions are equivalent:*

$$\begin{cases} i) & \forall\, x \;\in\; X, \;\; \mathbf{v}(x) \;>\; -\infty \\ ii) & \mathbf{v}(0) \;=\; 0 \\ iii) & (0,-1) \;\notin\; \mathcal{E}p(\mathbf{v}) \end{cases}$$

Proof — Assume that the epigraph of $\mathbf{v}$ is closed and pick $x \in X$. There exists a sequence of elements x_n converging to x such that

$$\lim_{n \to \infty} \mathbf{v}(x_n) \;=\; \liminf_{x' \to x} \mathbf{v}(x')$$

Hence, for any $\lambda > \liminf_{x' \to x} \mathbf{v}(x')$, there exist N such that, for all $n \ge N$, $\mathbf{v}(x_n) \le \lambda$, i.e., such that $(x_n, \lambda) \in \mathcal{E}p(\mathbf{v})$. By taking the limit, we infer that $\mathbf{v}(x) \le \lambda$, and thus, that $\mathbf{v}(x) \le \liminf_{x' \to x} \mathbf{v}(x')$. The converse statement is obvious.

Suppose next that the epigraph of $\mathbf{v}$ is a cone. Then it contains $(0,0)$ and $\mathbf{v}(0) \le 0$. The statements *ii*) and *iii*) are clearly equivalent.

If $i)$ holds true and $\mathbf{v}(0) < 0$, then

$$(0,-1) \;=\; \frac{1}{-\mathbf{v}(0)}(0,\mathbf{v}(0))$$

belongs to the epigraph of $\mathbf{v}$, as well as all $(0,-\lambda)$, and (by letting $\lambda \to +\infty$) we deduce that $\mathbf{v}(0) = -\infty$, so that $i)$ implies $ii)$.

To end the proof, assume that $\mathbf{v}(0) = 0$ and that for some x, $\mathbf{v}(x) = -\infty$. Then, for any $\varepsilon > 0$, the pair $(x, -1/\varepsilon)$ belongs to the epigraph of $\mathbf{v}$, as well as the pairs $(\varepsilon x, -1)$. By letting ε converge to 0, we infer that $(0,-1)$ belongs also to the epigraph, since it is closed. Hence $\mathbf{v}(0) < 0$, a contradiction. $\square$

Indicators ψ_K *of subsets* K are cost functions defined by

$$\psi_K(x) := 0 \;\text{ if }\; x \in K \;\text{ and }\; +\infty \;\text{ if not}$$

which characterize subsets (as *characteristic functions* do for other purposes) provide important examples of extended functions. It can be regarded as a *membership cost*[1] to K: it costs nothing to belong to K, and $+\infty$ to step outside of K.

Since

$$\mathcal{E}p(\psi K) \;=\; K \times \mathbf{R}_+$$

we deduce that the indicator ψ_K is lower semicontinuous if and only if K is closed and that ψ_K is convex if and only if K is convex. One can regard the sum $\mathbf{v} + \psi_K$ as the restriction of $\mathbf{v}$ to K.

We recall the convention $\inf(\emptyset) := +\infty$.

3.2 Epilimits

Definition 3.2.1 *The epigraph of the* lower epilimit $\lim^\sharp_{\uparrow n\to\infty}\mathbf{u}_n$ *of a sequence of extended functions* $\mathbf{u}_n : X \mapsto \mathbf{R}\cup\{+\infty\}$ *is the upper limit of the epigraphs:*

$$\mathcal{E}p(\lim^\sharp_{\uparrow n\to\infty}\mathbf{u}_n) \;:=\; \mathrm{Limsup}_{n\to\infty}\mathcal{E}p(\mathbf{u}_n)$$

The function $\lim^\flat_{\uparrow n\to\infty}\mathbf{u}_n$ *whose epigraph is the lower limit of the epigraphs of the functions* $\mathbf{u}_n$

$$\mathcal{E}p(\lim^\flat_{\uparrow n\to\infty}\mathbf{u}_n) \;:=\; \mathrm{Liminf}_{n\to\infty}\mathcal{E}p(\mathbf{u}_n)$$

is the upper epilimit *of the functions* $\mathbf{u}_n$

One can check that

$$\lim^\sharp_{\uparrow n\to\infty}\mathbf{u}_n(x_0) \;=\; \liminf_{n\to\infty, x\to x_0} \mathbf{u}_n(x)$$

[1]Functions $\mathbf{v} : X \mapsto [0,+\infty]$ can be regarded as some kind of *fuzzy sets*, called *toll sets*.

3.3 Contingent Epiderivatives

When $\mathbf{u}$ is an extended function, we associate with it its epigraph and the contingent cones to this epigraph. This leads to the concept of epiderivatives of extended functions.

Definition 3.3.1 *Let* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ *be a nontrivial extended function and* x *belong to its domain.*

We associate with it the differential quotients

$$u \rightsquigarrow \nabla_h \mathbf{u}(x)(u) := \frac{\mathbf{u}(x+hu) - \mathbf{u}(x)}{h}$$

The contingent epiderivative $D_\uparrow \mathbf{u}(x)$ *of* $\mathbf{u}$ *at* $x \in \mathrm{Dom}(\mathbf{u})$ *is the lower epilimit of its differential quotients:*

$$D_\uparrow \mathbf{u}(x) = \lim{}^{\sharp}_{\uparrow h \to 0+} \nabla_h \mathbf{u}(x)$$

We shall say that the function $\mathbf{u}$ *is* contingently epidifferentiable *at* x *if for any* $u \in X$, $D_\uparrow \mathbf{u}(x)(u) > -\infty$ *(or, equivalently, if* $D_\uparrow \mathbf{u}(x)(0) = 0$*).*

Proposition 3.3.2 *Let* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ *be a nontrivial extended function and* x *belong to its domain. Then the contingent epiderivative* $D_\uparrow \mathbf{u}(x)$ *satisfies*

$$\forall\, u \in X, \quad D_\uparrow \mathbf{u}(x)(u) = \liminf_{h \to 0+, u' \to u} \frac{\mathbf{u}(x+hu') - \mathbf{u}(x)}{h}$$

and the epigraph of the contingent epiderivative $D_\uparrow \mathbf{u}(\cdot)$ *is equal to the contingent cone to the epigraph of* $\mathbf{u}$ *at* $(x, \mathbf{u}(x))$ *is*

$$\mathcal{E}p(D_\uparrow \mathbf{u}(x)) = T_{\mathcal{E}p(\mathbf{u})}(x, \mathbf{u}(x))$$

Proof — The first statement is obvious. For proving the second one, we recall that the contingent cone

$$T_{\mathcal{E}p(\mathbf{u})}(x, \mathbf{u}(x)) = \mathrm{Limsup}_{h \to 0+} \frac{\mathcal{E}p(\mathbf{u}) - (x, \mathbf{u}(x))}{h}$$

is the upper limit of the differential quotients $\dfrac{\mathcal{E}p(\mathbf{u}) - (x, \mathbf{u}(x))}{h}$ when $h \to 0+$. It is enough to observe that

$$\mathcal{E}p(D_\uparrow \mathbf{u}(x)) := T_{\mathcal{E}p(\mathbf{u})}(x, y) \ \ \& \ \ \mathcal{E}p(\nabla_h \mathbf{u}(x)) = \frac{\mathcal{E}p(\mathbf{u}) - (x, \mathbf{u}(x))}{h}$$

to conclude. □

Consequently, *the epigraph of the contingent epiderivative at x is a closed cone. It is then lower semicontinuous and positively homogeneous whenever $\mathbf{u}$ is contingently epidifferentiable at x.*

We observe that the contingent epiderivative of the indicator function ψ_K at $x \in K$ is the indicator of the contingent cone to K at x:

$$D_\uparrow \psi_K(x) \ = \ \psi_{T_K(x)}$$

making precise the intuition stating that the contingent cone $T_K(x)$ plays the role of a "derivative of a set", as the limit of differential quotients $\frac{K-x}{h}$ of sets.

The hypoderivatives of an extended function are defined in a analogous way: The contingent hypoderivative $D_\downarrow \mathbf{u}(x)$ of $\mathbf{u}$ at $x \in \mathrm{Dom}(\mathbf{u})$ is the upper hypolimit of its differential quotients:

$$D_\downarrow \mathbf{u}(x) \ = \ \lim{}^\sharp_{\downarrow}{}_{h \to 0+} \nabla_h \mathbf{u}(x)$$

We observe that it is equal to

$$\forall \, u \in X, \ \ D_\downarrow \mathbf{u}(x)(u) \ = \ \limsup_{h \to 0+, u' \to u} \frac{\mathbf{u}(x+hu') - \mathbf{u}(x)}{h}$$

and that *the hypograph of the* contingent hypoderivative $D_\downarrow \mathbf{u}(x)$ *of* $\mathbf{u}$ *at* x *is the contingent cone to the hypograph of* $\mathbf{u}$ *at* $(x, \mathbf{u}(x))$:

$$\mathcal{E}p(D_\downarrow \mathbf{u}(x)) = T_{\mathcal{H}yp(\mathbf{u})}(x, \mathbf{u}(x))$$

Definition 3.3.3 *We shall say that* $\mathbf{u} : X \mapsto W$ *is* differentiable from the right *at* x *if the contingent epiderivative and hypoderivative coincide:*

$$\forall \, v \in X, \ \ D_\uparrow \mathbf{u}(x)(v) \ = \ D_\downarrow \mathbf{u}(x)(v)$$

Lemma 3.3.4 *Let* $K \subset X$ *be a closed subset and* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *be an extended function. We denote by* $\mathbf{u}|_K := f + \psi_K$ *the restriction to* $\mathbf{u}$ *at* K*. Inequality*

$$D_\uparrow \mathbf{u}(x)|_{T_K}(x) \ \le \ D_\uparrow \mathbf{u}|_K(x)$$

always holds true. It is an equality when $\mathbf{u}$ *is differentiable from the right:* the contingent derivative of the restriction of $\mathbf{u}$ to K is the restriction of the derivative to the contingent cone.

Proof — Indeed, let $x \in K \cap \mathrm{Dom}(\mathbf{u})$. If u belongs to $T_K(x)$, there exist $h_n \to 0+$, $\varepsilon_n \to 0+$ and $x_n := x + h_n u_n \in K$ such that

$$D_\uparrow \mathbf{u}(x)(u) \ \le \ \liminf_{n\to+\infty} \frac{\mathbf{u}(x_n) - \mathbf{u}(x)}{h_n} \ = \ \liminf_{n\to+\infty} \frac{\mathbf{u}|_K(x_n) - \mathbf{u}|_K(x)}{h_n}$$

which implies the inequality. If $\mathbf{u}$ is differentiable from the right, the differential quotient converges to the common value $D_\uparrow \mathbf{u}(x) = D_\uparrow \mathbf{u}|_K(x) = D_\downarrow \mathbf{u}|_K(x)$. □

For locally Lipschitz functions, the contingent epiderivatives are finite:

Proposition 3.3.5 *If* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *is Lipschitz around* $x \in \mathrm{Int}(\mathrm{Dom}(\mathbf{u}))$, *then the contingent epiderivative* $D_\uparrow \mathbf{u}(x)$ *is Lipschitz: there exists* $\lambda > 0$ *such that*

$$\forall\, u \in X, \ \ D_\uparrow \mathbf{u}(x)(u) \ = \ \liminf_{h\to 0+} \frac{\mathbf{u}(x+hu) - \mathbf{u}(x)}{h} \ \le \ \lambda \|u\|$$

Proof — Since $\mathbf{u}$ is Lipschitz on some ball $B(x,\eta)$, the above inequality follows immediately from

$$\forall\, u \in \eta B, \ \ \frac{\mathbf{u}(x+hu) - \mathbf{u}(x)}{h} \ \le \ \frac{\mathbf{u}(x+hu') - \mathbf{u}(x)}{h} + \lambda(\|u\| + \|u'-u\|)$$

by taking the liminf when $h \to 0+$ and $u' \to u$. □

For convex functions, we obtain:

Proposition 3.3.6 *When the function* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *is convex, the contingent epiderivative is equal to*

$$D_\uparrow \mathbf{u}(x)(u) \ = \ \liminf_{u'\to u} \left(\inf_{h>0} \frac{\mathbf{u}(x+hu') - \mathbf{u}(x)}{h} \right)$$

Proof — Indeed, Proposition 1.2.5 implies that if $0 < h_1 \le h_2$,

$$\mathcal{E}p(\nabla_{h_2} \mathbf{u}(x)) \ = \ \frac{\mathcal{E}p(\mathbf{u}) - (x, \mathbf{u}(x))}{h_2} \ \subset \ \mathcal{E}p(\nabla_{h_1} \mathbf{u}(x))$$

i.e.,

$$\forall\, u \in X, \ \ \nabla_{h_1} \mathbf{u}(x)(u) \ \le \ \nabla_{h_2} \mathbf{u}(x)(u)$$

Therefore,

$$\forall\, u \in X, \ \ D\mathbf{u}(x)(u) \ := \ \lim_{h\to 0+} \frac{\mathbf{u}(x+hu) - \mathbf{u}(x)}{h} \ = \ \inf_{h>0} \frac{\mathbf{u}(x+hu) - \mathbf{u}(x)}{h}$$

and this function $D\mathbf{u}(x)$ is convex with respect to u. Since the epigraph of $D\mathbf{u}(x)$ is the increasing union of the epigraphs of the differential quotients $\nabla_h\mathbf{u}(x)$, we infer that

$$D_\uparrow\mathbf{u}(x)(u) \;:=\; \liminf_{u'\to u} D\mathbf{u}(x)(u')$$

We recall the following important property of convex functions defined on finite dimensional vector spaces:

Theorem 3.3.7 *An extended convex function $\mathbf{u}$ defined on a finite dimensional vector-space is locally Lipschitz and subdifferentiable on the interior of its domain. Therefore, when x belongs to the interior of the domain of $\mathbf{u}$, there exists a constant λ_x such that*

$$\forall\, u \;\in\; X,\;\; D_\uparrow\mathbf{u}(x)(u) \;=\; \inf_{h>0}\frac{\mathbf{u}(x+hu)-\mathbf{u}(x)}{h} \;\leq\; \lambda_x\|u\|$$

The second statement follows from Proposition 3.3.5. □

3.4 Generalized Gradients

Definition 3.4.1 *Let $\mathbf{u}: X \mapsto \mathbf{R}\cup\{+\infty\}$ be a nontrivial extended function. The continuous linear functionals $p\in X^\star$ satisfying*

$$\forall\, v\in X,\;\; \langle p,v\rangle \;\leq\; D_\uparrow\mathbf{u}(x)(v)$$

are called the (regular) subgradients *of $\mathbf{u}$ at x, which constitute the (possibly empty) closed convex subset*

$$\partial_-\mathbf{u}(x) \;:=\; \{p\in X^\star \mid \forall\, v\in X,\;\; \langle p,v\rangle \;\;\leq\; D_\uparrow\mathbf{u}(x)(v)\}$$

called the (regular) subdifferential *of $\mathbf{u}$ at x_0.*

In a symmetric way, the superdifferential $\partial_+\mathbf{u}(x)$ *of $\mathbf{u}$ at x is defined by*

$$\partial_+\mathbf{u}(x) \;:=\; -\partial_-(-\mathbf{u})(x)$$

Naturally, when $\mathbf{u}$ is Fréchet differentiable at x, then

$$D_\uparrow\mathbf{u}(x)(v) \;=\; \langle f'(x),v\rangle$$

so that *the subdifferential $\partial_-\mathbf{u}(x)$ is reduced to the gradient $\mathbf{u}'(x)$.*

We observe that

$$\partial_-\mathbf{u}(x)+N_K(x) \;\subset\; \partial(\mathbf{u}|_K)(x)$$

If $\mathbf{u}$ is differentiable at a point $x \in K$, then the *subdifferential of the restriction is the sum of the gradient and the normal cone*:

$$\partial_-(\mathbf{u}|_K)(x) \; = \; \mathbf{u}'(x) + N_K(x)$$

We also note that the subdifferential of the indicator of a subset is the normal cone:

$$\partial_-\psi_K(x) \; = \; N_K(x)$$

that

$$\begin{cases} i) & (p,-1) \in N_{\mathcal{E}p(\mathbf{u})}(x,\mathbf{u}(x)) \text{ if and only if } p \in \partial_-\mathbf{u}(x) \\ ii) & (p,0) \in N_{\mathcal{E}p(\mathbf{u})}(x,\mathbf{u}(x)) \text{ if and only if } p \in \mathrm{Dom}(D_\uparrow \mathbf{u}(x))^- \end{cases}$$

so that We also deduce that

$$N_{\mathcal{E}p(\mathbf{u})}(x,\mathbf{u}(x)) \; = \; \{\lambda(q,-1)\}_{q\in\partial_-\mathbf{u}(x),\,\lambda>0} \bigcup \{(q,0)\}_{q\in\mathrm{Dom}(D_\uparrow\mathbf{u}(x))^-}$$

The subset $\mathrm{Dom}(D_\uparrow \mathbf{u}(x))^- = \{0\}$ whenever the domain of the contingent epiderivative $D_\uparrow \mathbf{u}(x)$ is dense in X. This happens when $\mathbf{u}$ is locally Lipschitz and when the dimension of X is finite:

Proposition 3.4.2 *Let X be a finite dimensional vector space, $\mathbf{u} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be a nontrivial extended function and $x_0 \in \mathrm{Dom}(\mathbf{u})$. Then the subdifferential $\partial_-\mathbf{u}(x)$ is the set of elements $p \in X^\star$ satisfying*

$$\liminf_{x\to x_0} \frac{\mathbf{u}(x) - \mathbf{u}(x_0) - \langle p, x - x_0\rangle}{\|x - x_0\|} \; \geq \; 0 \qquad (3.1)$$

is the local subdifferential of $\mathbf{u}$ at x_0.

In a symmetric way, the superdifferential $\partial_+\mathbf{u}(x_0)$ *of* $\mathbf{u}$ *at* x_0 *is the subset of elements* $p \in X^\star$ *satisfying*

$$\limsup_{x\to x_0} \frac{\mathbf{u}(x) - \mathbf{u}(x_0) - \langle p, x - x_0\rangle}{\|x - x_0\|} \; \leq \; 0$$

Proof — This is an easy consequence of proposition 1.2.10. □

The equivalent formulation (3.1) of the concept of subdifferential has been introduced by Crandall & P.-L. Lions for defining *viscosity solutions* to Hamilton-Jacobi equations.

3.5 Moreau-Rockafellar Subdifferentials

When $\mathbf{u}$ is convex, the generalized gradient coincides with the subdifferential introduced by Moreau and Rockafellar for convex functions in the early 60's:

Definition 3.5.1 *Consider a nontrivial function* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *and* $x \in \mathrm{Dom}(\mathbf{u})$. *The closed convex subset* $\partial\mathbf{u}(x)$ *defined by*

$$\partial\mathbf{u}(x) \ = \ \{p \ \in \ X^\star \mid \forall\, y \ \in \ X, \ \langle p, y - x\rangle \ \leq \ \mathbf{u}(y) - \mathbf{u}(x)\}$$

(which may be empty) is called the Moreau-Rockafellar subdifferential *of* $\mathbf{u}$ *at* x. *We say that* $\mathbf{u}$ *is* subdifferentiable at x *if* $\partial\mathbf{u}(x) \neq \emptyset$.

Proposition 1.2.9 implies that in the convex case

Proposition 3.5.2 *Let* $\mathbf{u} : X \mapsto \mathbf{R}_+$ *be a nontrivial extended convex function. Then the subdifferential* $\partial_-\mathbf{u}(x)$ *coincides with Moreau-Rockafellar subdifferential* $\partial\mathbf{u}(x)$.

Furthermore, the graph of the subdifferential map $x \rightsquigarrow \partial\mathbf{u}(x)$ *is closed.*

Let us mention the following simple — but useful — remark:

Proposition 3.5.3 *Assume that* $\mathbf{u} := \mathbf{v} + \mathbf{w}$ *is the sum of a differentiable function* $\mathbf{v}$ *and a convex function* $\mathbf{w}$. *If* $\bar{x}$ *minimizes* $\mathbf{u}$, *then*

$$-\mathbf{v}'(\bar{x}) \ \in \ \partial\mathbf{w}(\bar{x})$$

Proof — Indeed, for $h > 0$ small enough, $\bar{x} + h(y - \bar{x}) = (1-h)\bar{x} + hy$ so that

$$0 \ \leq \ \frac{\mathbf{u}(\bar{x} + h(y-\bar{x})) - \mathbf{u}(\bar{x})}{h} \ \leq \ \frac{\mathbf{u}(\bar{x} + h(y-\bar{x})) - \mathbf{u}(\bar{x})}{h} + \mathbf{w}(y) - \mathbf{w}(\bar{x})$$

thanks to the convexity of $\mathbf{w}$. Letting h converge to 0 yields

$$0 \ \leq \ \langle \mathbf{v}'(\bar{x}), y - \bar{x}\rangle + \mathbf{w}(y) - \mathbf{w}(\bar{x})$$

so that $-\mathbf{v}'(\bar{x})$ belongs to $\partial\mathbf{w}(\bar{x})$. $\square$

Chapter 4

Some Hamilton-Jacobi Equations

Introduction

Let us introduce

1. a differential equation $x' = f(x)$, where $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is continuous and has linear growth,

2. a nonnegative continuous "Lagrangian"

$$l : (x,p) \in \mathbf{R}^n \times \mathbf{R}^n \mapsto l(x,p) \in \mathbf{R}_+$$

3. an extended nonnegative lower semicontinuous function $\mathbf{u} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$

We consider the problem

$$\mathbf{u}^\top(x) \;:=\; \alpha^\top_{(f,l)}(\mathbf{u})(x) \;:=\; \inf_{x(\cdot)\in\mathcal{S}_f(x)} \left(\sup_{t\geq 0} \left(e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau))d\tau \right) \right)$$

and the "stopping time" problem

$$\mathbf{u}^\perp(x) \;:=\; \alpha^\perp_{(f,l)}(\mathbf{u})(x) \;:=\; \inf_{x(\cdot)\in\mathcal{S}_f(x)} \left(\inf_{t\geq 0} \left(e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau))d\tau \right) \right)$$

whenever the graph of the function $\mathbf{u}$ is regarded as an obstacle, as in unilateral mechanics. Taking $l \equiv 0$, we obtain the a-Lyapunov function

$$\alpha^{\top}_{(f,0)}(\mathbf{u})(x) \ := \ \inf_{x(\cdot)\in\mathcal{S}_f(x)} \left(\sup_{t\geq 0} e^{at}\mathbf{u}(x(t)) \right)$$

as an example of the first problem and taking $\mathbf{u} \equiv 0$, we obtain the variational problem

$$\alpha^{\top}_{(f,l)}(0)(x) \ := \ \inf_{x(\cdot)\in\mathcal{S}_f(x)} \int_0^{\infty} e^{a\tau} l(x(\tau), x'(\tau))d\tau$$

We shall prove that these functions are "generalized" solutions to Hamilton-Jacobi "differential variational inequalities"

$$\begin{cases} i) & \mathbf{u}(x) \ \leq \ \mathbf{u}^{\top}(x) \\ ii) & \left\langle \dfrac{\partial}{\partial x}\mathbf{u}^{\top}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\top}(x) \ \leq \ 0 \\ iii) & (\mathbf{u}(x) - \mathbf{u}^{\top}(x)) \left(\left\langle \dfrac{\partial}{\partial x}\mathbf{u}^{\top}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\top}(x) \right) \ = \ 0 \end{cases}$$

and

$$\begin{cases} i) & 0 \ \leq \ \mathbf{u}^{\perp}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \left\langle \dfrac{\partial}{\partial x}\mathbf{u}^{\perp}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\perp}(x) \ \geq \ 0 \\ iii) & (\mathbf{u}(x) - \mathbf{u}^{\perp}(x)) \left(\left\langle \dfrac{\partial}{\partial x}\mathbf{u}^{\perp}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\perp}(x) \right) \ = \ 0 \end{cases}$$

respectively.

For this type of partial differential equations, the concept of distributional derivatives happens to be much less adequate than other concepts of "contingent epiderivatives", as it was shown by Hélène Frankowska, or subdifferentials, as they appear in the concept of "viscosity solutions" introduced by Michael Crandall and Pierre-Louis Lions for a general class of nonlinear Hamilton-Jacobi equations.

Indeed, for those Hamilton-Jacobi type equations derived from the calculus of variations, control theory or differential games, one can derive from the properties of the viability kernels and capture basins of auxiliary systems that the various value functions involved in these problems are solutions to such Hamilton-Jacobi equations or differential variational inequalities in an adequate sense.

We illustrate here this general approach for the two preceding problems only. Namely, we shall prove that the epigraphs of the two "value functions" $\mathbf{u}^{\top}$ and $\mathbf{u}^{\perp}$ are respectively the viability kernel and the capture basin of the epigraph of the function $\mathbf{u}$ under the map $g : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n \times \mathbf{R}$ defined by

$$g(x, y) \ := \ (f(x), -ay - l(x, f(x)))$$

which is a continuous map with linear growth whenever f and the Lagrangian l are continuous with linear growth. The basic observation allowing this transfer of properties is summarized in the statement of the following:

Lemma 4.0.4 *If a closed subset $\mathcal{V} \subset \mathbf{R}^n \times \mathbf{R}_+$ is locally viable under g, so is the epigraph of the associated extended function $\mathbf{v} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by*

$$\mathbf{v}(x) \ := \ \inf_{(x,y) \in \mathcal{V}} y$$

where we set as usual $\mathbf{v}(x) := +\infty$ whenever the subset $\{y \mid (x,y) \in \mathcal{V}\}$ is empty. As a consequence, for any $x \in \mathrm{Dom}(\mathbf{v})$, there exist a solution to the differential equation $x' = f(x)$ and $T > 0$ satisfying

$$\forall\, t \in [0,T], \ \ e^{at}\mathbf{v}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau)) d\tau \ \leq \ \mathbf{v}(x) \tag{4.1}$$

(where $T = +\infty$ whenever $\mathcal{V}$ is (globally) viable under g).

Proof — If x belongs to the domain of $\mathbf{v}$, then $(x, \mathbf{v}(x))$ belongs to $\mathcal{V}$ so that there exist a solution $x(\cdot) \in \mathcal{S}_f(x)$ and $T > 0$ such that

$$\forall\, t \in [0,T], \ \ \left(x(t), y(t) := e^{-at}\mathbf{v}(x) - \int_0^t e^{-a(t-\tau} l(x(\tau), x'(\tau)) d\tau\right) \ \in \ \mathcal{V}$$

i.e., if and only if $y(t) \geq \mathbf{v}(x(t))$, which can be written in the form (4.1). If $y_0 > \mathbf{v}(x)$, then we observe that for all $t \geq 0$,

$$y_0(t) \ := \ e^{-at} y_0 - \int_0^t e^{-a(t-\tau} l(x(\tau), x'(\tau)) d\tau \ \geq \ y(t) \ \geq \ \mathbf{v}(x(t))$$

and thus, that $(x(t), y_0(t))$ is a solution to the differential equation $(x'(t), y'(t)) = g(x(t), y(t))$ starting at (x, y_0) and viable in the epigraph of $\mathbf{v}$. $\square$

4.1 Value Function

We associate with $\mathbf{u} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$ the problem

$$\mathbf{u}^\top(x) \ := \ \inf_{x(\cdot) \in \mathcal{S}_f(x)} \left(\sup_{t \geq 0} \left(e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau)) d\tau \right) \right)$$

The function $\mathbf{u}^\top := \alpha^\top_{(f,l)}(\mathbf{u})$ is called the value function associated with $\mathbf{u}$.

If $l = 0$, the above problem can be written

$$\alpha^\top_{(f,0)}(\mathbf{u})(x) \ := \ \inf_{x(\cdot) \in \mathcal{S}_f(x)} \left(\sup_{t \geq 0} e^{at}\mathbf{u}(x(t)) \right)$$

and if $\mathbf{u} \equiv 0$, the above problem boils down to

$$\alpha^{\top}_{(f,l)}(0)(x) \;:=\; \inf_{x(\cdot)\in\mathcal{S}_f(x)} \int_0^{\infty} e^{a\tau} l(x(\tau), x'(\tau))d\tau$$

Before investigating further these examples, we begin by characterizing the epigraph of $\mathbf{u}^{\top}$:

Proposition 4.1.1 *Let us assume that f and l are continuous with linear growth and that $\mathbf{u} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is nontrivial, non negative and lower semicontinuous.*

Then the epigraph of $\mathbf{u}^{\top} := \alpha^{\top}_{(f,l)}(\mathbf{u})$ is the viability kernel $\mathrm{Viab}_g(\mathcal{E}p(\mathbf{u}))$ of the epigraph of $\mathbf{u}$ under g.

Consequently, the function $\mathbf{u}^{\top}$ is characterized as the smallest of the lower semicontinuous functions $\mathbf{v} : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ larger than or equal to $\mathbf{u}$ such that for any $x \in \mathrm{Dom}(\mathbf{v})$, there exists a solution $x(\cdot)$ to the differential equation $x' = f(x)$ satisfying property (4.1):

$$\forall\, t \geq 0,\;\; e^{at}\mathbf{v}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau))d\tau \;\leq\; \mathbf{v}(x)$$

Proof — Indeed, to say that a pair (x, y) belongs to the viability kernel $\mathrm{Viab}_g(\mathcal{E}p(\mathbf{u}))$ means that there exists a solution $x(\cdot) \in \mathcal{S}_f(x)$ such that

$$\forall\, t \geq 0,\;\; \left(x(t), e^{-at}y - \int_0^t e^{-a(t-\tau} l(x(\tau), x'(\tau))d\tau\right) \;\in\; \mathcal{E}p(\mathbf{u})$$

i.e., if and only if

$$\forall\, t \geq 0,\;\; e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau))d\tau \;\leq\; y$$

This implies that

$$\mathbf{u}^{\top}(x) \;:=\; \inf_{x(\cdot)\in\mathcal{S}_f(x)} \left(\sup_{t\geq 0}\left(\mathbf{u}(x(t)) + \int_0^t l(x(\tau), x'(\tau))d\tau\right)\right) \;\leq\; y$$

and thus, that $\mathrm{Viab}_g(\mathcal{E}p(\mathbf{u}))$ is contained in $\mathcal{E}p(\mathbf{u}^{\top})$.

Since the set $\mathcal{S}_f(x)$ of solutions is compact in the space $\mathcal{C}(0, \infty; \mathbf{R}^n)$ thanks to Theorem 1.5.2 and since the function $x(\cdot) \mapsto \int_0^t l(x(\tau), f(x(\tau)))$ is continuous on $\mathcal{C}(0, \infty; \mathbf{R}^n)$, the infimum

$$\mathbf{u}^{\top}(x) \;:=\; \sup_{t\geq 0}\left(e^{at}\mathbf{u}(\bar{x}(t)) + \int_0^t e^{a\tau} l(\bar{x}(\tau), \bar{x}'(\tau))d\tau\right)$$

is reached by a solution $\bar{x}(\cdot) \in \mathcal{S}_f(x)$. Consequently, the function

$$\left(\bar{x}(t), e^{-at}\mathbf{u}^{\top}(x) - \int_0^t e^{-a(t-\tau}l(\bar{x}(\tau), \bar{x}'(\tau))d\tau\right) \in \mathcal{E}p(\mathbf{u}^{\top})$$

is viable in the epigraph of $\mathbf{u}^{\top}$. Therefore, $(x, \mathbf{u}^{\top}(x))$ belongs to the viability kernel of the epigraph of $\mathbf{u}$.

By Lemma 4.0.4, the epigraph of $\mathbf{u}^{\top}$ being the viability kernel, contains the epigraph of any lower semicontinuous functions $\mathbf{v} : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ larger than or equal to $\mathbf{u}$ viable under g, i.e., satisfying property (4.1). Therefore, the function $\mathbf{u}^{\top}$ is the smallest of the lower semicontinuous functions $\mathbf{v} : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ larger than or equal to $\mathbf{u}$ satisfying property (4.1). □

Theorem 4.1.2 *We posit the assumptions of Theorem 4.1.1. Then the value function $\mathbf{u}^{\top}$ is characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ satisfying for every x*

$$\begin{cases} i) & \mathbf{u}(x) \leq \mathbf{v}(x) \\ ii) & D_{\uparrow}\mathbf{v}(x)(f(x)) + l(x, f(x)) + a\mathbf{v}(x) \leq 0 \end{cases}$$

Furthermore, it satisfies the property

$$\begin{cases} \forall\, x \quad \textit{such that}\, \mathbf{u}(x) < \mathbf{u}^{\top}(x), \\ D_{\uparrow}\mathbf{u}^{\top}(\mathbf{u})(x)(-f(x)) - l(x, f(x)) - a\mathbf{u}^{\top}(x) \leq 0 \end{cases}$$

Remark — If the function $\mathbf{u}^{\top}$ is differentiable, then the contingent epiderivative coincides with the usual derivatives, so that $\mathbf{u}^{\top}$ is a solution to the linear Hamilton-Jacobi "differential variational inequalities"

$$\begin{cases} i) & \mathbf{u}(x) \leq \mathbf{u}^{\top}(x) \\ ii) & \left\langle \frac{\partial}{\partial x}\mathbf{u}^{\top}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\top}(x) \leq 0 \\ iii) & (\mathbf{u}(x) - \mathbf{u}^{\top}(x))\left(\left\langle \frac{\partial}{\partial x}\mathbf{u}^{\top}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\top}(x)\right) = 0 \end{cases}$$

Proof — By the Nagumo Theorem, the epigraph of $\mathbf{v}$ is viable under g if and only if

$$\forall\, (x, y) \in \mathcal{E}p(\mathbf{v}), \quad (f(x), -ay - l(x, f(x))) \in T_{\mathcal{E}p(\mathbf{v})}(x, y)$$

When $y = \mathbf{v}(x)$, we deduce from the fact that the contingent cone to the epigraph of $\mathbf{v}$ at $(x, \mathbf{v}(x))$

$$T_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x)) := \mathcal{E}p(D_{\uparrow}\mathbf{v}(x))$$

that if the epigraph of $\mathbf{v}$ is viable under g, then

$$\forall\, x \in \mathrm{Dom}(\mathbf{v}),\;\; D_{\uparrow}\mathbf{v}(x)(f(x)) + l(x, f(x)) + a\mathbf{v}(x) \;\leq\; 0$$

Conversely, this inequality implies that $(f(x), -a\mathbf{v}(x) - l(x, f(x))) \in T_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x))$. It also implies that if $y > \mathbf{v}(x)$, $(f(x), -ay - l(x, f(x)))$ belongs to $T_{\mathcal{E}p(\mathbf{v})}(x, y)$. Indeed, we know that there exist sequences $h_n > 0$ converging to 0, v_n converging to $f(x)$ and ε_n converging to 0 such that

$$(x + h_n v_n, \mathbf{v}(x) - h_n(a\mathbf{v}(x) + l(x, f(x)) + h_n\varepsilon_n) \;\in\; \mathcal{E}p(\mathbf{v})$$

We thus deduce that

$$\begin{cases} (x + h_n v_n, y - h_n(ay + l(x, f(x)) + h_n\varepsilon_n) \\ = \;(x + h_n v_n, \mathbf{v}(x) - h_n(a\mathbf{v}(x) + l(x, f(x)) + h_n\varepsilon_n) + (0, (1 - h_n)(y - \mathbf{v}(x))) \\ \in\; \mathcal{E}p(\mathbf{v}) + \{0\} \times \mathbf{R}_+ \;=\; \mathcal{E}p(\mathbf{v}) \end{cases}$$

and thus, that $(f(x), -ay - l(x, f(x)))$ belongs to $T_{\mathcal{E}p(\mathbf{v})}(x, y)$.

Finally, Theorem 2.3.1 states that $\mathbf{u}^{\top}$ satisfies

$$\forall\, (x, \mathbf{u}^{\top}(x)) \in \mathcal{E}p(\mathbf{u}^{\top})\backslash\partial\,(\mathcal{E}p(\mathbf{u}))\,,\;\; (-f(x), a\mathbf{u}^{\top}(x) + l(x, f(x))) \;\in\; T_{\mathcal{E}p(\mathbf{u}^{\top})}(x, \mathbf{u}^{\top}(x)) \tag{4.2}$$

which, joined to the other properties, can be translated as the "Frankowska" solution to the variational inequalities. □

4.2 Lyapunov Functions

Consider a differential equation $x' = f(x)$ and a nontrivial nonnegative lower semicontinuous extended function $\mathbf{u} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$.

The function $\alpha^{\top}_{(f,0)}(\mathbf{u}) : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\alpha^{\top}_{(f,0)}(\mathbf{u})(x) \;:=\; \inf_{x(\cdot)\in\mathcal{S}_f(x)} \left(\sup_{t\geq 0} e^{at}\mathbf{u}(x(t))\right)$$

is said *to enjoy the a-Lyapunov property* because for any initial state x_0, there exists a solution to the differential equation $x' = f(x)$ satisfying

$$\forall\, t \geq 0,\;\; \mathbf{u}(x(t)) \;\leq\; e^{-at}\alpha^{\top}_{(f,0)}(\mathbf{u})(x_0) \tag{4.3}$$

Such inequalities allow us to deduce many properties on the asymptotic behavior of $\mathbf{v}$ along the solutions to the differential equation. This may be quite useful when $\mathbf{u}$ is the distance function $d_M(\cdot)$ to a subset. The domain of this Lyapunov function

$\alpha^{\top}_{(f,0)}(d_M)$ provides the a-basin of attraction of M, which is the set of states from which a solution $x(\cdot)$ to the differential equation converges exponentially to M:

$$\forall\, x_0 \in \mathrm{Dom}(\alpha^{\top}_{(f,0)}(d_M)),\;\; d_M(x(t)) \;\le\; e^{-at}\alpha^{\top}_{(f,0)}(d_M)(x_0)$$

The main question we face is *to characterize this Lyapunov function*. Ever since Lyapunov proposed in 1892 his second method for studying the behavior of a solution around an equilibrium, finding Lyapunov functions for such and such differential equation has been a source of numerous problems requiring most often many clever tricks.

We deduce from Theorems 4.1.1 and 4.1.2 with $l = 0$ the following characterization of Lyapunov functions:

Theorem 4.2.1 *Let us assume that f is continuous with linear growth and that $\mathbf{u} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is nontrivial, non negative and lower semicontinuous. Then the epigraph of the Lyapunov function $\alpha^{\top}_{(f,0)}(\mathbf{u})$ is the viability kernel* $\mathrm{Viab}_g(\mathcal{E}p(\mathbf{u}))$ *of the epigraph of $\mathbf{u}$ under g. Therefore, the Lyapunov function $\alpha^{\top}_{(f,0)}(\mathbf{u})$ is the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ enjoying the a-Lyapunov property, i.e., such that from any $x_0 \in \mathrm{Dom}(\mathbf{v})$ starts at least one solution to the differential equation $x' = f(x)$ satisfying*

$$\forall\, t \ge 0,\;\; \mathbf{v}(x(t)) \;\le\; \mathbf{v}(x_0)e^{-at}$$

or, equivalently,

$$D_{\uparrow}\mathbf{v}(x)(f(x)) + a\mathbf{v}(x) \;\le\; 0$$

Furthermore, if $\mathbf{u}(x) < \alpha^{\top}_{(f,0)}(\mathbf{u})(x)$, it satisfies

$$D_{\uparrow}\alpha^{\top}_{(f,0)}(\mathbf{u})(x)(-f(x)) - a\alpha^{\top}_{(f,0)}(\mathbf{u})(x) \;\le\; 0$$

4.3 Finite Length Solutions

We define now $l(x,p) := \|p\|$, so that $l(x, f(x)) = \|f(x)\|$, and take $a := 0$ and $\mathbf{u}(x) := 0$. Then

$$\alpha^{\top}_{(f,\|f\|)}(0)(x_0) \;=\; \inf_{x(\cdot)\in\mathcal{S}_f(x)} \int_0^{+\infty} \|x'(\tau)\| d\tau$$

is the minimal length of the trajectories of the solutions $x(\cdot)$ to the differential equation $x' = f(x)$ starting from x_0.

Its epigraph is the viability kernel of $\mathbf{R}^n \times \mathbf{R}_+$ under the system of differential equations $(x', y') = (f(x), -\|f(x)\|)$. The minimal length is the smallest of the

nonnegative lower semicontinuous functions $\mathbf{v} : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ satisfying for every x

$$D_{\uparrow}\mathbf{v}(x)(f(x)) + \|f(x)\| \leq 0$$

and satisfies whenever the length $\alpha^{\top}_{f,\|f\|}(0)(x) > 0$ is strictly positive

$$D_{\uparrow}\alpha^{\top}_{f,\|f\|}(0)(x)(-f(x)) - \|f(x)\| \leq 0$$

4.4 Stopping Time Problem

We still consider the $\mathbf{u} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$, regarded as an "obstacle" in problems of unilateral mechanics. We associate with it the stopping time problem

$$\mathbf{u}^{\perp}(x) := \alpha^{\perp}_{(f,l)}(\mathbf{u})(x) := \inf_{x(\cdot)\in\mathcal{S}_f(x)} \left(\inf_{t\geq 0}\left(e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau))d\tau\right)\right)$$

We begin by characterizing its epigraph:

Proposition 4.4.1 *Let us assume that f and l are continuous with linear growth and that $\mathbf{u} : \mathbf{R}^n \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is nontrivial, non negative and lower semicontinuous.*

Then the epigraph of $\mathbf{u}^{\perp} := \alpha^{\perp}_{(f,l)}(\mathbf{u})$ is the capture basin $\mathrm{Capt}_g(\mathcal{E}p(\mathbf{u}))$ of the epigraph of $\mathbf{u}$ under g.

Proof — To say that a pair (x, y) belongs to the capture basin $\mathrm{Capt}_g(\mathcal{E}p(\mathbf{u}))$ means that there exist a solution $(x(\cdot)) \in \mathcal{S}_{(f)}(x)$ and $t \geq 0$ such that

$$\left(x(t), e^{-at}y - \int_0^t e^{-a(t-\tau}l(x(\tau), x'(\tau))d\tau\right) \in \mathcal{E}p(\mathbf{u})$$

i.e., if and only if

$$e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau}l(x(\tau), x'(\tau))d\tau \leq y$$

This implies that

$$\begin{cases} \mathbf{u}^{\perp}(x) \\ := \inf_{x(\cdot)\in\mathcal{S}_f(x)} \left(\inf_{t\geq 0}\left(e^{at}\mathbf{u}(x(t)) + \int_0^t e^{a\tau}l(x(\tau), x'(\tau))d\tau\right)\right) \leq y \end{cases}$$

and thus, that $\mathrm{Capt}_g(\mathcal{E}p(\mathbf{u}))$ is contained in $\mathcal{E}p(\mathbf{u}^{\perp})$.

Since the infimum

$$\mathbf{u}^{\perp}(x) := e^{a\bar{t}}\mathbf{u}(\bar{x}(\bar{t})) + \int_0^{\bar{t}} e^{a\tau}l(\bar{x}(\tau), \bar{x}'(\tau))d\tau$$

is reached by a solution $\bar{x}(\cdot) \in \mathcal{S}_f(x)$ at a time $\bar{t}$, this states that $(x, \mathbf{u}^{\perp}(x))$ belongs to the capture basin of the epigraph of $\mathbf{u}$. □

In order to apply the properties of capture basins, we need to check that $X \times \mathbf{R}_+$ is a repeller under the auxiliary system g.

Lemma 4.4.2 *Let us assume that there exist real constants γ_- and δ^- such that*

$$\begin{cases} i) & \inf_{x \in X} \frac{\langle x, f(x) \rangle}{\|x\|} \geq \gamma_-(\|x\|+1) \\ ii) & \inf_{x \in X} l(x, f(x)) \geq \delta_-(\|x\|+1) \end{cases}$$

and if $a + \gamma_- > 0$, then $X \times \mathbf{R}_+$ is a repeller.

Proof — Whenever l is nonnegative, the backward solutions $(x(\cdot), y(\cdot))$ starting from $X \times \mathbf{R}_+$ are viable in $X \times \mathbf{R}_+$ because $y'(t) = -ay(t) + l(x(t)) \geq 0$.

Let $(x(\cdot), y(\cdot))$ be the solution to the differential equation $(x', y') = g(x, y)$ starting from (x_0, y_0).

Therefore

$$\frac{d}{dt}\|x(t)\| = \left\langle x'(t), \frac{x(t)}{\|x(t)\|} \right\rangle = \left\langle f(x(t))), \frac{x(t)}{\|x(t)\|} \right\rangle \geq \gamma_-(\|x(t)\|+1)$$

so that

$$\forall\, t \geq 0, \ \ \|x(t)\| \geq e^{\gamma_- t}(\|x_0\|+1) - 1$$

Furthermore, since

$$l(x(\tau), x'(\tau)) \geq \delta_-(\|x(\tau)\|+1) \geq \delta_-(\|x_0\|+1)e^{\gamma_- t}$$

and since

$$e^{at}y(t) = y_0 - \int_0^t e^{a\tau} l(x(\tau)) d\tau$$

we infer that

$$e^{at}y(t) \leq y_0 - \delta_-(\|x_0\|+1) \int_0^t e^{\gamma_- + a)\tau} = y_0 - \frac{\delta_-(\|x_0\|+1)}{\gamma_- + a}\left(e^{(\gamma_- + a)t} - 1\right)$$

Consequently, if $\gamma - + a > 0$

$$e^{at}y(t) \leq y_0 + \frac{\delta_-(\|x_0\|+1)}{\gamma_- + a} - \frac{\delta_-(\|x_0\|+1)}{\gamma_- + a} e^{(\gamma_- - a)t}$$

so that $y(t)$ becomes negative in finite time. □

Theorem 4.4.3 *We posit the assumptions of Proposition 4.4.1 and we assume that*

$$\forall\, x \in K,\ \ \forall\, x(\cdot) \in \mathcal{S}_f(x),\ \ \int_0^{+\infty} e^{a\tau} l(x(\tau), x'(\tau))d\tau \ = \ +\infty$$

Then $\mathbf{u}^{\perp}$ *is characterized as the* **unique** *nonnegative lower semicontinuous functions* $\mathbf{v} : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ *such that from any* x *satisfying* $\mathbf{v}(x) < \mathbf{u}(x)$ *starts a solution* $x(\cdot) \in \mathcal{S}_f(x)$ *satisfying, for some time* $T > 0$

$$\forall\, t \in [0, T],\ \ e^{at}\mathbf{v}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau))d\tau \ \leq \ \mathbf{v}(x)$$

and that, for any $T > 0$ *and any* $x_T \in \mathrm{Dom}(\mathbf{u})$*, all solutions* $x(\cdot)$ *to the differential equation* $x' = f(x)$ *arriving at* x_T *at time* T *satisfy*

$$\forall\, t \in [0, T],\ \ e^{at}\mathbf{v}(x(t)) + \int_0^t e^{a\tau} l(x(\tau), x'(\tau))d\tau \ \leq \ \mathbf{v}(x(0))$$

The function $\mathbf{u}^{\perp}$ *is also the smallest of the lower semicontinuous functions* $\mathbf{v}$ *satisfying*

$$\begin{cases} i) & 0 \ \leq \ \mathbf{v}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \text{if } 0 \leq \mathbf{v}(x) < \mathbf{u}(x),\ \ D_{\uparrow}\mathbf{v}(x)(f(x)) + l(x, f(x)) + a\mathbf{v}(x) \ \leq \ 0 \end{cases}$$

If we assume furthermore that f *and* l *are Lipschitz, then the function* $\mathbf{u}^{\perp}$ *is the* **unique** *solution* $\mathbf{v} \geq 0$ *to the system of "differential inequalities": for every* $x \in \mathrm{Dom}(\mathbf{v})$*,*

$$\begin{cases} i) & 0 \ \leq \ \mathbf{v}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \text{if } 0 \leq \mathbf{v}(x) < \mathbf{u}(x),\ \ D_{\uparrow}\mathbf{v}(x)(f(x)) + l(x) + a\mathbf{v}(x) \ \leq \ 0 \\ iii) & \text{if } 0 \leq \mathbf{v}(x) \leq \mathbf{u}(x),\ \ D_{\uparrow}\mathbf{v}(x)(-f(x) - l(x), f(x)) - a\mathbf{v}(x) \ \leq \ 0 \end{cases}$$

Knowing the function $\mathbf{u}^{\perp}$*, the stopping time is the first time* $\bar{t} \geq 0$ *when* $\mathbf{v}(x(\bar{t})) = \mathbf{u}^{\perp}(x(\bar{t}))$*.*

Remark — If the function $\mathbf{u}^{\perp} := \alpha^{\perp}_{(f,l)}(\mathbf{u})$ is differentiable, then the contingent epiderivative coincides with the usual derivatives, so that $\mathbf{u}^{\perp}$ is a solution to the linear Hamilton-Jacobi "differential variational inequalities"

$$\begin{cases} i) & 0 \ \leq \ \mathbf{u}^{\perp}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \left\langle \dfrac{\partial}{\partial x}\mathbf{u}^{\perp}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\perp}(x) \ \geq \ 0 \\ iii) & (\mathbf{u}(x) - \mathbf{u}^{\perp}(x)) \left(\left\langle \dfrac{\partial}{\partial x}\mathbf{u}^{\perp}(x), f(x) \right\rangle + l(x, f(x)) + a\mathbf{u}^{\perp}(x) \right) \ = \ 0 \end{cases}$$

Proof — Since the Lagrangian is nonnegative, the closed subset $\mathbf{R}^n \times \mathbf{R}_+$ is backward invariant under g. It is a repeller under g whenever

$$\forall\, x \in \mathbf{R}^n,\ \forall\, x(\cdot) \in \mathcal{S}_f(x),\ \int_0^{+\infty} e^{a\tau} l(x(\tau), x'(\tau))d\tau \;=\; +\infty$$

By Theorem 2.5.1, $\mathcal{E}p(\mathbf{u}^{\perp}) := \mathrm{Capt}_g(\mathcal{E}p(\mathbf{u}))$ is the unique closed subset $\mathbf{v}$, and in particular, the unique epigraph $\mathcal{E}p(\mathbf{v})$ of lower semicontinuous function $\mathbf{v}$, which satisfies

$$\begin{cases} i) & \mathcal{E}p(\mathbf{u}) \;\subset\; \mathcal{E}p(\mathbf{v}) \;\subset\; \mathbf{R}^n \times \mathbf{R}_+ \\ ii) & \mathcal{E}p(\mathbf{v})\backslash\mathcal{E}p(\mathbf{u}) \text{ is locally viable under } g \\ iii) & \mathcal{E}p(\mathbf{v}) \text{ is backward invariant under } g \end{cases}$$

The first condition means that for any $x \in \mathbf{R}^n$, $0 \le \mathbf{v}(x) \le \mathbf{u}(x)$ and we observe that $(x, y) \in \mathcal{E}p(\mathbf{v})\backslash\mathcal{E}p(\mathbf{u})$ if and only if $y \in [\mathbf{v}(x), \mathbf{u}(x)[$. Hence, the first statement of the theorem ensues.

By Theorem 2.4.1, $\mathcal{E}p(\mathbf{u}^{\perp}) := \mathrm{Capt}_g(\mathcal{E}p(\mathbf{u}))$ is also the smallest of the nonnegative lower semicontinuous $\mathbf{v}$ satisfying

$$\begin{cases} i) & \mathcal{E}p(\mathbf{u}) \;\subset\; \mathcal{E}p(\mathbf{v}) \;\subset\; \mathbf{R}^n \times \mathbf{R}_+ \\ ii) & \forall\,(x, y) \in \mathcal{E}p(\mathbf{v})\backslash\mathcal{E}p(\mathbf{u}),\ \ (f(x), -l(x, f(x)) - ay) \;\in\; T_{\mathcal{E}p(\mathbf{v})}(x, y) \end{cases}$$

When $y = \mathbf{v}(x)$, the second condition can be written

$$\text{if } \mathbf{v}(x) < \mathbf{u}(x),\ \ D_{\uparrow}\mathbf{v}(x) + a\mathbf{v}(x) + l(x, f(x)) \;\le\; 0$$

Conversely, this condition implies that for any $y \in]\mathbf{v}(x), \mathbf{u}(x)[$, $(f(x), -l(x, f(x)) - ay)$ also belongs to $T_{\mathcal{E}p(\mathbf{v})}(x, y)$ as in the proof of Theorem 4.1.2.

Since $\mathcal{E}p(\mathbf{u}^{\perp}) = \mathrm{Capt}_g(\mathcal{E}p(\mathbf{u}))$ is the unique closed subset satisfying the above properties and being backward invariant under g, this implies that

$$\text{if } \mathbf{v}(x) \le \mathbf{u}(x),\ \ D_{\uparrow}\mathbf{v}(x)(-f(x) - l(x), f(x)) - a\mathbf{v}(x) \;\le\; 0$$

The converse is true when f and l are Lipschitz, or whenever the solution to the system $(x', y') = g(x, y)$ is unique. In this case, the function $\mathbf{u}^{\perp}$ is the unique solution satisfying the two properties. $\square$

4.5 Minimal Time and Minimal Length Solutions

Let us consider a closed subset $K \subset \mathbf{R}^n$ and ψ_K its indicator and take $a = 0$.

We observe that the hitting time (or minimal time) function $\omega_K^{f^\flat}$ is equal to

$$\omega_K^{f^\flat} \;=\; \alpha_{f,1}^{\perp}(\psi_K)$$

In the same way, we introduce the *minimal length functional* associating with $x(\cdot)$

$$\lambda_K(x(\cdot)) \ := \inf_{\{t \mid x(t)\in K\}} \int_0^t \|x'(s)\|ds$$

the minimal length of the curve $s \mapsto x(s)$ from 0 to t such that $x(t) \in K$. We next define the "minimal length" function $\lambda_K^{f^\flat}$ by

$$\lambda_K^{f^\flat}(x) \ := \inf_{x(\cdot)\in\mathcal{S}_f(x)} \lambda_K(x(\cdot))$$

We note that

$$\lambda_K^{f^\flat} \ = \ \alpha^{\perp}_{f,\|f\|}(\psi_K)$$

So, these two functions enjoy the properties proved above. For instance:

1. The minimal time function $\omega_K^{f^\flat}$ is the smallest nonnegative lower semicontinuous $\mathbf{v}$ function vanishing on K such that,

$$\forall\, x \notin K, \ \ D_{\uparrow}\mathbf{v}(x)(f(x)) + 1 \ \leq \ 0$$

If f is assumed furthermore to be Lipschitz, it is the **unique** nonnegative lower semicontinuous solution vanishing on K satisfying the above inequalities and

$$\forall\, x \in \mathbf{R}^n, \ \ D_{\uparrow}\mathbf{v}(x)(-f(x)) - 1 \ \leq \ 0$$

2. Assume that

$$\forall\, x \in \mathbf{R}^n, \ \ \inf_{x(\cdot)\in\mathcal{S}_f(x)} \int_0^{+\infty} \|x'(s)\|ds \ = \ +\infty$$

Then the minimal length function $\lambda_K^{f^\flat}$ is the smallest nonnegative lower semicontinuous $\mathbf{v}$ function vanishing on K such that,

$$\forall\, x \notin K, \ \ D_{\uparrow}\mathbf{v}(x)(f(x)) + \|f(x)\| \ \leq \ 0$$

If f is assumed furthermore to be Lipschitz, it is the **unique** nonnegative lower semicontinuous solution vanishing on K satisfying the above inequalities and

$$\forall\, x \in \mathbf{R}^n, \ \ D_{\uparrow}\mathbf{v}(x)(-f(x)) - \|f(x)\| \ \leq \ 0$$

Since the minimal time and minimal length functions coincide with the indicator of ψ_K on K, the above conditions imply that K is backward invariant under f whenever f is Lipschitz.

4.6 Viscosity Type Solutions

We now use the characterizations in terms of normal cones for deriving the formulations in terms of subgradients instead of contingent epiderivatives.

Theorem 4.6.1 *We posit the assumptions of Theorem 4.1.1. Then the value function* $\mathbf{u}^\top$ *is the solution to*

$$\left\{\begin{array}{ll} i) & \mathbf{u}(x) \;\leq\; \mathbf{u}^\top(x) \\ ii) & \forall\, p \in \partial_-\mathbf{u}^\top(x),\;\; \langle p, f(x)\rangle + l(x, f(x)) + a\mathbf{u}^\top(x) \;\leq\; 0 \\ & \text{and} \\ & \forall\, p \in \mathrm{Dom}(D_\uparrow \mathbf{u}^\top(x))^-,\;\; \langle p, f(x)\rangle \;\leq 0 \\ iii) & \forall\, p \in \partial_-\mathbf{u}^\top(x),\;\; (\mathbf{u}(x) - \mathbf{u}^\top(x))\left(\langle p, f(x)\rangle + l(x, f(x)) + a\mathbf{u}^\top(x)\right) \;=\; 0 \\ & \text{and} \\ & \forall\, p \in \mathrm{Dom}(D_\uparrow \mathbf{u}^\top(x))^-,\;\; (\mathbf{u}(x) - \mathbf{u}^\top(x))\langle p, f(x)\rangle \;= 0 \end{array}\right.$$

Such a solution, recently discovered independently by Frankowska and Barron & Jensen, are sometime called "bilateral solutions" to Hamilton-Jacobi equation

$$\left\langle \frac{\partial}{\partial x}\mathbf{u}(x), f(x)\right\rangle + l(x, f(x)) + a\mathbf{u}(x) \;=\; 0$$

The method we present here is due to Frankowska.

Proof — By Theorem 4.1.2, we know that whenever $\mathbf{u}(x) \leq \mathbf{u}^\top(x)$,

$$(f(x), -a\mathbf{u}^\top(x) - l(x, f(x))) \;\in\; T_{\mathcal{E}p(\mathbf{u}^\top)}(x, \mathbf{u}^\top(x))$$

and that whenever $\mathbf{u}(x) < \mathbf{u}^\top(x)$,

$$(f(x), -a\mathbf{u}^\top(x) - l(x, f(x))) \;\in\; T_{\mathcal{E}p(\mathbf{u}^\top)}(x, \mathbf{u}^\top(x)) \cap -T_{\mathcal{E}p(\mathbf{u}^\top)}(x, \mathbf{u}^\top(x))$$

Theorem 1.2.11 implies that these conditions are equivalent to

$$\forall\, (p, \lambda) \in N_{\mathcal{E}p(\mathbf{u}^\top)}(x, \mathbf{u}^\top(x)),\;\; \langle p, f(x)\rangle - \lambda(a\mathbf{u}^\top(x) + l(x, f(x))) \;\leq\; 0$$

whenever $\mathbf{u}(x) \leq \mathbf{u}^\top(x)$, and to

$$\forall\, (p, \lambda) \in N_{\mathcal{E}p(\mathbf{u}^\top)}(x, \mathbf{u}^\top(x)),\;\; \langle p, f(x)\rangle - \lambda(a\mathbf{u}^\top(x) + l(x, f(x))) \;=\; 0$$

whenever $\mathbf{u}(x) < \mathbf{u}^\top(x)$.

It remains now to recall that

$$\left\{\begin{array}{ll} i) & (p, -1) \in N_{\mathcal{E}p(\mathbf{u}^\top)}(x, \mathbf{u}^\top(x)) \text{ if and only if } p \in \partial_-\mathbf{u}^\top(x) \\ ii) & (p, 0) \in N_{\mathcal{E}p(\mathbf{u}^\top)}(x, \mathbf{u}^\top(x)) \text{ if and only if } p \in \mathrm{Dom}(D_\uparrow \mathbf{u}^\top(x))^- \end{array}\right.$$

Taking $\lambda = -1$, we obtain

$$\forall\, p \in \partial_- \mathbf{u}^\top(x), \;\; \langle p, f(x)\rangle - \lambda(a\mathbf{u}^\top(x) + l(x, f(x))) \;\leq\; 0$$

and whenever $\mathbf{u}(x) < \mathbf{u}^\top(x)$,

$$\forall\, p \in \partial_- \mathbf{u}^\top(x), \;\; \langle p, f(x)\rangle - \lambda(a\mathbf{u}^\top(x) + l(x, f(x))) \;=\; 0$$

Taking $\lambda = 0$ yields that for all $p \in \mathrm{Dom}(D_\uparrow \mathbf{u}^\top(x))^-$, $\langle p, f(x)\rangle \leq 0$ if $\mathbf{u}(x) \leq \mathbf{u}^\top(x)$ and $\langle p, f(x)\rangle = 0$ if $\mathbf{u}(x) < \mathbf{u}^\top(x)$. This means that $f(x)$ belongs to the closure of $\mathrm{Dom}(D_\uparrow \mathbf{u}^\top(x))$ in the general case and that x belongs to the vector space spanned by $\mathrm{Dom}(D_\uparrow \mathbf{u}^\top(x))$ when $\mathbf{u}(x) < \mathbf{u}^\top(x)$. $\square$

We obtain an analogous statement for the function $\mathbf{u}^\perp$:

Theorem 4.6.2 *We posit the assumptions of Theorem 4.4.3. Then the value function $\mathbf{u}^\perp$ is the solution to*

$$\begin{cases} i) & 0 \;\leq\; \mathbf{u}^\perp(x) \;\leq\; \mathbf{u}(x) \\ ii) & \forall\, p \in \partial_- \mathbf{u}^\perp(x), \;\; \langle p, f(x)\rangle + l(x, f(x)) + a\mathbf{u}^\perp(x) \;\geq\; 0 \\ & \text{and} \\ & \forall\, p \in \mathrm{Dom}(D\mathbf{u}_\downarrow^\top(x))^-, \;\; \langle p, f(x)\rangle \;\geq 0 \\ iii) & \forall\, p \in \partial_- \mathbf{u}^\perp(x), \;\; (\mathbf{u}(x) - \mathbf{u}^\perp(x))\Big(\langle p, f(x)\rangle + l(x, f(x)) + a\mathbf{u}^\perp(x)\Big) \;=\; 0 \\ & \text{and} \\ & \forall\, p \in \mathrm{Dom}(D_\downarrow \mathbf{u}^\top(x))^-, \;\; (\mathbf{u}(x) - \mathbf{u}^\top(x))\langle p, f(x)\rangle \;= 0 \end{cases}$$

Proof — By Theorem 4.4.3, we know that whenever $\mathbf{u}(x) \geq \mathbf{u}^\perp(x)$,

$$(-f(x), a\mathbf{u}^\perp(x) + l(x, f(x))) \;\in\; T_{\mathcal{E}p(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x))$$

and that whenever $\mathbf{u}(x) > \mathbf{u}^\perp(x)$,

$$(f(x), -a\mathbf{u}^\perp(x) - l(x, f(x))) \;\in\; T_{\mathcal{E}p(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x)) \cap -T_{\mathcal{E}p(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x))$$

Theorem 1.2.11 implies that these conditions are equivalent to whenever $\mathbf{u}(x) \geq \mathbf{u}^\perp(x)$,

$$\forall\, (p, \lambda) \in N_{\mathcal{E}p(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x)), \;\; \langle p, f(x)\rangle - \lambda(a\mathbf{u}^\perp(x) + l(x, f(x))) \;\geq\; 0$$

and that whenever $\mathbf{u}(x) > \mathbf{u}^\perp(x)$,

$$\forall\, (p, \lambda) \in N_{\mathcal{E}p(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x)), \;\; \langle p, f(x)\rangle - \lambda(a\mathbf{u}^\perp(x) + l(x, f(x))) \;=\; 0$$

It remains now to translate these statements in terms of subgradients. $\square$

Remark: Viscosity Solutions — When we know *a priori* that the solution $\mathbf{u}^\perp$ is continuous, we can prove that it is also a “viscosity solution” the the Hamilton-Jacobi variational inequalities:

Theorem 4.6.3 *We posit the assumptions of Theorem 4.4.3, and we assume that f and l are Lipschitz and the function $\mathbf{u}^\perp$ is continuous. Then the value function $\mathbf{u}^\perp$ is the solution to*

$$\left\{\begin{array}{ll} i) & 0 \leq \mathbf{u}^\perp(x) \leq \mathbf{u}(x) \\ ii) & \forall\, p \in \partial_+ \mathbf{u}^\perp(x),\ \ \langle -p, f(x)\rangle + l(x, f(x)) + a\mathbf{u}^\perp(x) \ \geq\ 0 \\ & \\ & \text{and} \\ & \forall\, p \in \mathrm{Dom}(D_\downarrow \mathbf{u}^\perp)^-,\ \ \langle p, f(x)\rangle\ \leq 0 \\ iii) & \forall\, p \in \partial_- \mathbf{u}^\perp(x),\ \ \langle p, f(x)\rangle + l(x, f(x)) + a\mathbf{u}^\perp(x) \ \leq\ 0 \\ & \text{and} \\ & \forall\, p \in \mathrm{Dom}(D_\uparrow \mathbf{u}^\perp(x))^-,\ \ \langle p, f(x)\rangle\ \leq 0 \end{array}\right.$$

A solution to such a system of inequalities is called a 'viscosity solution' to Hamilton-Jacobi equation

$$\left\langle \frac{\partial}{\partial x}\mathbf{u}(x), f(x)\right\rangle + l(x, f(x)) + a\mathbf{u}^\perp(x) \ =\ 0$$

by Michael Crandall and Pierre-Louis Lions.

Proof — We know that epigraph of $\mathbf{u}^\perp$ is backward invariant under g. By Lemmas 1.7.2 and 1.7.3, this implies that its complement is forward invariant under g, and, since f and l are Lipschitz, that its closure is also forward invariant. Since we assumed that $\mathbf{u}^\perp$ is continuous, the closure of the complement of the epigraph of $\mathbf{u}^\perp$ is the hypograph of $\mathbf{u}^\perp$. Therefore,

$$\forall\, (x, \mathbf{u}^\perp(x)) \in \mathrm{Graph}(\mathbf{u}^\perp),\ \ (f(x), -a\mathbf{u}^\perp(x) - l(x, f(x)))\ \in\ T_{\mathcal{H}yp(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x))$$

and thus,

$$\forall\, (p, \lambda) \in N_{\mathcal{H}yp(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x)),\ \ \langle p, f(x)\rangle - \lambda(a\mathbf{u}^\perp(x) + l(x, f(x)))\ \leq\ 0$$

It remains now to recall that

$$\left\{\begin{array}{ll} i) & (p,1) \in N_{\mathcal{H}yp(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x)) \text{ if and only if } p \in \partial_+\mathbf{u}^\perp(x) \\ ii) & (p,0) \in N_{\mathcal{H}yp(\mathbf{u}^\perp)}(x, \mathbf{u}^\perp(x)) \text{ if and only if } \forall\, p \in \mathrm{Dom}(D_\downarrow \mathbf{u}^\perp(x))^- \end{array}\right.$$

for deducing that the above condition is equivalent to

$$\forall\, p \in \partial_+\mathbf{u}^\perp(x),\ \ \langle -p, f(x)\rangle + a\mathbf{u}^\perp(x) + l(x, f(x))\ \geq 0$$

and that $f(x)$ belongs to the closure of the domain of $D_\downarrow \mathbf{u}^\perp(x)$ for achieving the proof. □

Chapter 5

Systems of First-Order Partial Differential Equations

Introduction

We study here Dirichlet boundary value problems for systems of first-order partial differential equations of the form

$$\forall\, j = 1, \ldots, p, \quad \frac{\partial}{\partial t} u(t,x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} u_i(t,x) f_i(t,x,u(t,x)) - g_j(t,x,u(t,x)) \;=\; 0$$

on $\mathbf{R}_+ \times \Omega$, where $\Omega \subset \mathbf{R}^n$ is an open subset, Γ its boundary and $K := \overline{\Omega}$ its closure.

It is known that the solution U to the above system can be set-valued, describing "shocks". This is considered as a pathology whenever the solution is regarded as a map from the input space $\mathbf{R}_+ \times K$ to the output space $\mathbf{R}^p$, but is quite natural when the solution U is considered as a graph, i.e., a subset of $\mathbf{R}_+ \times K \times \mathbf{R}^p$ and when the tools of set-valued analysis are used. Since we are looking for set-valued map solutions, we begin by introducing

1. the graph of the *upper graphical limit* of a sequence of maps $U_n : X \rightsquigarrow Y$ (single-valued or set-valued) is the upper limit of the graphs of U_n,

2. *the contingent derivative* $DU(x,y)$ at point (x,y) of the graph of U is the upper graphical limit of the difference quotients $\nabla_h U(x,y)$, so that the graph of the contingent derivative is the contingent cone to the graph of U:

$$\text{Graph}(DU(x,y)) \;=\; T_{\text{Graph}(U)(x,y)}$$

Introducing

1. an *initial data* $u_0 : K \mapsto \mathbf{R}^p$,

2. a *boundary data* $v_\Gamma : \mathbf{R}_+ \times \partial K \mapsto \mathbf{R}^p$.

we shall prove the *existence and the uniqueness* of a set-valued solution : $\mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ to the system of first-order partial differential equations satisfying the initial/boundary-value conditions

$$\begin{cases} i) & \forall\, x \in K,\;\; u_0(x) \;\in\; U(0,x) \\ ii) & \forall\, t > 0,\; \forall\, x \in \Gamma,\;\; v_\Gamma(t,x) \;\in\; U(t,x) \end{cases} \tag{5.1}$$

Again, the strategy is the same than the one we followed for Hamilton-Jacobi variational inequalities. It is enough to revive the method of characteristics by observing that the graph of the solution is the capture basin of the graph of the initial/boundary-value data under an auxiliary system (the "characteristic system"), use the characterizations derived from the Nagumo Theorem and the fact that the contingent cone to the graph is the graph of the contingent derivative of a set-valued map.

However, the solution becomes single-valued when the maps f_i depend only on the variables x. In this case, we even obtain explicit formulas.

5.1 Contingent Derivatives of Set-Valued Maps

5.1.1 Set-Valued Maps

Definition 5.1.1 *Let X and Y be two spaces. A set-valued map F from X to Y is characterized by its* graph *$Graph(F)$, the subset of the product space $X \times Y$ defined by*

$$\text{Graph}(F) \;:=\; \{(x,y) \in X \times Y \mid\; y \in F(x)\}$$

We shall say that $F(x)$ is the image *or the* value *of F at x.*

A set-valued map is said to be nontrivial *if its graph is not empty, i.e., if there exists at least an element $x \in X$ such that $F(x)$ is not empty.*

We say that F is strict *if all images $F(x)$ are not empty. The* domain *of F is the subset of elements $x \in X$ such that $F(x)$ is not empty:*

$$\mathrm{Dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$$

The image *of F is the union of the images (or values) $F(x)$, when x ranges over X:*

$$\mathrm{Im}(F) := \bigcup_{x \in X} F(x)$$

The inverse *F^{-1} of F is the set-valued map from Y to X defined by*

$$x \in F^{-1}(y) \iff y \in F(x) \iff (x, y) \in \mathrm{Graph}(F)$$

We shall emphasize the characterization of a set-valued map (as well as a single-valued map) by its graph. This point of view has been coined the *graphical approach* by R.T. Rockafellar.

The domain of F is thus the image of F^{-1} and coincides with the projection of the graph onto the space X and, in a symmetric way, the image of F is equal to the domain of F^{-1} and to the projection of the graph of F onto the space Y.

Sequences of subsets can be regarded as set-valued maps defined on the set $\mathbf{N}$ of integers.

5.1.2 Graphical Convergence of Maps

Since the graphical approach consists in regarding closed set-valued maps as graphs, i.e., as closed subsets of the product space, ranging over the space $\mathcal{F}(X \times Y)$, one can supply this space with upper and lower limits, providing the concept of upper and lower graphical convergence:

Definition 5.1.2 *Let us consider metric spaces X, Y and a sequence of set-valued maps $F_n : X \leadsto Y$. The set-valued maps $\mathrm{Lim}^\sharp{}_{n\to\infty} F_n$ and $\mathrm{Lim}^\flat{}_{n\to\infty} F_n$ from X to Y defined by*

$$\begin{cases} i) & \mathrm{Graph}(\mathrm{Lim}^\sharp{}_{n\to\infty} F_n) := \mathrm{Limsup}_{n\to\infty} \mathrm{Graph}(F_n) \\ \\ ii) & \mathrm{Graph}(\mathrm{Lim}^\flat{}_{n\to\infty} F_n) := \mathrm{Liminf}_{n\to\infty} \mathrm{Graph}(F_n) \end{cases}$$

are called the (graphical) upper and lower limits *of the set-valued maps F_n respectively.*

Even for single-valued maps, this is a weaker convergence than the pointwise convergence:

Proposition 5.1.3

1. *If $f_n : X \mapsto Y$ converges pointwise to f, then, for every $x \in X$, $f(x) \in f^\sharp(x)$. If the sequence is equicontinuous, then $f^\sharp(x) = \{f(x)\}$.*

2. *Let $\Omega \subset \mathbf{R}^n$ be an open subset. If a sequence $f_n \in L^p(\Omega)$ converges to f in $L^p(\Omega)$, then*
$$\text{for almost all } x \in \Omega, \quad f(x) \in f^\sharp(x)$$

5.1.3 Contingent Derivatives

Let $F : X \rightsquigarrow Y$ be a set-valued map. We introduce the *differential quotients*

$$u \rightsquigarrow \nabla_h F(x,y)(u) \ := \ \frac{F(x+hu)-y}{h}$$

of a set-valued map $F : X \rightsquigarrow Y$ at $(x,y) \in \text{Graph}(F)$.

Definition 5.1.4 *The contingent derivative $DF(x,y)$ of F at $(x,y) \in \text{Graph}(F)$ is the graphical upper limit of differential quotients:*

$$DF(x,y) \ := \ \text{Lim}^\sharp{}_{h\to 0+} \nabla_h F(x,y)$$

In other words, v belongs to $DF(x,y)(u)$ if and only if there exist sequences $h_n \to 0^+$, $u_n \to u$ and $v_n \to v$ such that $\forall n \geq 0, \ \ y + h_n v_n \in F(x + h_n u_n)$.

In particular, if $f : X \mapsto Y$ is a single valued function, we set $Df(x) = Df(x, f(x))$.

We deduce the fundamental formula on the graph of the contingent derivative:

Proposition 5.1.5 *The graph of the contingent derivative of a set-valued map is the contingent cone to its graph: for all $(x,y) \in \text{Graph}(F)$,*

$$\text{Graph}(DF(x,y)) = T_{\text{Graph}(F)}(x,y)$$

Proof — Indeed, we know that the contingent cone

$$T_{\text{Graph}(F)}(x,y) \ = \ \text{Limsup}_{h\to 0+} \frac{\text{Graph}(F) - (x,y)}{h}$$

is the upper limit of the differential quotients $\frac{\text{Graph}(F)-(x,y)}{h}$ when $h \to 0+$. It is enough to observe that

$$\text{Graph}(\nabla_h F(x,y)) = \frac{\text{Graph}(F) - (x,y)}{h}$$

and to take the upper limit to conclude. $\square$

We can easily compute the derivative of the inverse of a set-valued map F (or even of a noninjective single-valued map): *The contingent derivative of the inverse of a set-valued map F is the inverse of the contingent derivative*:

$$D(F^{-1})(y,x) \quad = \quad DF(x,y)^{-1}$$

If K is a subset of X and f is a single-valued map which is Fréchet differentiable around a point $x \in K$, then *the contingent derivative of the restriction of f to K is the restriction of the derivative to the contingent cone*:

$$D(f|_K)(x) = D(f|_K)(x, f(x)) = f'(x)|_{T_K(x)}$$

5.2 Frankowska Solutions to First-Order Partial Differential Equations

We consider two finite dimensional vector spaces $\mathbf{R}^n$ and $\mathbf{R}^p$, an open subset $\Omega \subset \mathbf{R}^n$, its closure $K := \overline{\Omega}$ closed, its boundary $\Gamma := \partial\Omega = \Gamma$, two time-dependent maps $f : \mathbf{R}_+ \times K \times \mathbf{R}^p \mapsto \mathbf{R}^n$ and $g : \mathbf{R}_+ \times K \times \mathbf{R}^p \mapsto \mathbf{R}^p$.

We shall study the system of first-order partial differential equations

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}u(t,x)f(t,x,u(t,x)) - g(t,x,u(t,x)) \; = \; 0 \tag{5.2}$$

on $\mathbf{R}_+ \times K$.

It is known that the solution U to the above system can be set-valued, describing "shocks". This is considered as a pathology whenever the solution is regarded as a map from the input space $\mathbf{R}_+ \times K$ to the output space $\mathbf{R}^p$, but is quite natural when the solution U is considered as a graph, i.e., a subset of $\mathbf{R}_+ \times K \times \mathbf{R}^p$ and when the tools of set-valued analysis are used.

Introducing

1. an initial data $u_0 : K \mapsto \mathbf{R}^p$,

2. a boundary data $v_\Gamma : \mathbf{R}_+ \times \Gamma \mapsto \mathbf{R}^p$.

we shall prove the existence and the uniqueness of a solution : $\mathbf{R}_+ \times K \rightsquigarrow$ to the system of first-order partial differential equations (5.2) satisfying the initial/boundary-value conditions

$$\begin{cases} i) & \forall\, x \in K, \;\; u_0(x) \;\in\; U(0,x) \\ ii) & \forall\, t > 0, \; \forall\, x \in \Gamma, \;\; v_\Gamma(t,x) \;\in\; U(t,x) \end{cases} \tag{5.3}$$

Actually, we associate with the initial data $u_0 : K \mapsto \mathbf{R}^p$ and the boundary data $v_\Gamma : \mathbf{R}_+ \times \partial K \mapsto \mathbf{R}^p$ the "extended" boundary data $\Psi(u_0, v_\Gamma) : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ defined by

$$\Psi(u_0, v_\Gamma)(s,x) := \begin{cases} u_0(x) & \text{if} \quad s = 0 \quad \& \quad x \in K \\ v_\Gamma(s,x) & \text{if} \quad s \geq 0 \quad \& \quad x \in \partial K \\ \emptyset & \text{if} \quad s > 0 \quad \& \quad x \in \mathrm{Int}(K) \end{cases}$$

which is a set-valued map since it takes (empty) set values, the domain of which is $\mathrm{Dom}(\Psi(u_0, v_\Gamma)) := \partial(\mathbf{R}_+ \times K) = (\{0\} \times K) \cup (\mathbf{R}_+ \times \Gamma)$.

The set-valued map Ψ encapsulates or replaces initial/boundary-value data. Hence initial and boundary conditions (5.3) can be written in the form

$$\forall\, (t,x) \ \in \ \mathbf{R}_+ \times K, \ \ \Psi(u_0, v_\Gamma)(t,x) \ \subset \ U(t,x)$$

By the way, we can study as well the case when $\Psi : \mathbf{R}_+ \times \mathbf{R}^n \rightsquigarrow \mathbf{R}^p$ is any set-valued map, which allows to study other problems than initial/boundary-value problems associated with the system of first-order partial differential equations (5.2).

So, in the general case, we introduce two set-valued maps $\Psi : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ and $\Phi : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ satisfying

$$\forall\, (t,x) \ \in \ \mathbf{R}_+ \times K, \ \ \Psi(t,x) \ \subset \ \Phi(t,x)$$

We shall prove the existence and the uniqueness of a solution : $\mathbf{R}_+ \times K \rightsquigarrow$ to the system of first-order partial differential equations (5.2) satisfying the conditions

$$\forall\, (t,x) \in \mathbf{R}_+ \times K, \ \ \Psi(t,x) \ \subset \ U(t,x) \ \subset \ \Phi(t,x) \tag{5.4}$$

The set-valued map Φ describes "viability constraints" on the solution the solution U to the above system. The particular case without constraints is naturally obtained when $\Phi(t,x) := \mathbf{R}^p$.

Example: Impulse Boundary Value Problems This is the case when we provide boundary condition v_Γ^i only at impulse times t_i of an increasing sequence of impulse times $t_0 = 0 < t_1 \cdots < t_n < \cdots$.

We associate with them the map $\Psi : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ only derived from the initial data $u_0 : K \mapsto \mathbf{R}^p$ and the boundary data $v_{\partial K}^i$ by

$$\Psi(u_0, \{v_\Gamma\}^i)(s,x) := \begin{cases} u_0(x) & \text{if} \quad s = 0 & \& \quad x \in K \\ v_\Gamma^i(x) & \text{if} \quad s = t_i, \ i > 0, & \& \quad x \in \Gamma \\ \emptyset & \text{if} \quad s \in]t_i, t_{i+1}[, i > 0, & \& \quad x \in K \end{cases}$$

defined on $\mathrm{Dom}(\Psi(u_0, \{v_{\partial K}\}^i)) := (\{0\} \times K) \cup \bigcup_i (\{t_i\} \times \Gamma)$. $\square$

We denote by $h : \mathbf{R}_+ \times K \times \mathbf{R}^p \mapsto \mathbf{R}^n \mapsto \mathbf{R}_+ \times K \times \mathbf{R}^p \mapsto \mathbf{R}^n$ the map defined by

$$h(\tau, x, y) \;:=\; (1, f(\tau, x, y), g(\tau, x, y))$$

and the associated system $(\tau', x', y') = h(\tau, x, y)$ of differential equations

$$\left\{ \begin{array}{lrcl} i) & \tau'(t) & = & 1 \\ ii) & x'(t) & = & f(\tau(t), x(t), y(t)) \\ iii) & y'(t) & = & g(\tau(t), x(t), y(t)) \end{array} \right. \tag{5.5}$$

often called the associated "characteristic system".

Definition 5.2.1 *Given two time-dependent maps $f : \mathbf{R}_+ \times K \times \mathbf{R}^p \mapsto \mathbf{R}^n$ and $g : \mathbf{R}_+ \times K \times \mathbf{R}^p \mapsto \mathbf{R}^p$ and two set-valued maps $\Psi : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ and $\Phi : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ satisfying*

$$\forall\, (t, x) \;\in\; \mathbf{R}_+ \times K,\;\; \Psi(t, x) \;\subset\; \Phi(t, x)$$

we shall denote by $U := \mathcal{A}^{\Phi}_{(f,g)}(\Psi) : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ the set-valued map defined by

$$\mathrm{Graph}(U) \;:=\; \mathrm{Capt}_{-h}^{\mathrm{Graph}(\Phi)}(\mathrm{Graph}(\Psi)) \tag{5.6}$$

the graph of which is the viable-capture basin under $-h$ of the graph of Ψ in the graph of Φ.

Even when Ψ is single-valued on its domain, this map U can take several values, defined as "shocks" in the language of physicists.

Indeed, even though the trajectories of the solutions $(x(t), y(t))$ to the system of differential equation

$$\left\{ \begin{array}{lrcl} i) & x'(t) & = & f(t, x(t), y(t)) \\ ii) & y'(t) & = & g(t, x(t), y(t)) \end{array} \right.$$

never intersect in $\mathbf{R}^n \times \mathbf{R}^p$, their projections $x(t)$ onto $\mathbf{R}^n$ — privileged in his role of input space — may do so. In other words, if $x_1 \neq x_2$, the solutions $(x_i(t), y_i(t))$ starting from the initial conditions $(x_i, u_0(x_i))$ $(i = 1, 2)$ never intersect, but one cannot exclude the case when for some t, we may have $x_1(t) = x_2(t)$. At this time, the solution U takes (at least) the values $y_1(t) \neq y_2(t)$ associated with the common input $x := x_1(t) = x_2(t)$.

However, U is single-valued whenever $f(t, x, y) \equiv f(t, x)$ is independent of y and the above system has a unique solution for any initial condition.

Theorems 2.4.1 and 2.5.1 can be translated in terms of invariant manifold:

Definition 5.2.2 *We shall say that a set-valued map* $V : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ *defines an* invariant manifold *under the pair* (f, g) *if for every* $t_0 \geq 0$, $x_0 \in K$ *and* $y_0 \in V(t_0, x_0)$, *every solution* $(x(\cdot), y(\cdot))$ *to the system of differential equations*

$$\begin{cases} i) & x'(t) \;=\; f(t, x(t), y(t)) \\ ii) & y'(t) \;=\; g(t, x(t), y(t)) \end{cases}$$

starting at (x_0, y_0) *at time* t_0 *satisfies*

$$\forall\, t \geq t_0, \;\; y(t) \;\in\; V(t, x(t))$$

We observe that V defines an invariant manifold under (f, g) if and only if the graph of V is invariant under h.

Theorem 5.2.3 *Let us assume that the maps* f *and* g *are continuous with linear growth and that the graphs of the set-valued maps* $\Psi \subset \Phi$ *are closed. The associated set-valued map* $U := \mathcal{A}^{\Phi}_{(f,g)}(\Psi) : \mathbf{R}_+ \times K \rightsquigarrow \mathbf{R}^p$ *defined by (5.6) is the* **largest** *closed set-valued map enjoying the following properties:*

1. $\forall\, t \geq 0, \; \forall\, x \in K, \;\; \Psi(t, x) \;\subset\; U(t, x) \;\subset\; \Phi(t, x)$
2. *for every* t, x *and* $y \in U(t, x) \backslash \Psi(t, x)$, *there exist* $s \in [0, t[$, $x_s \in K$ *and* $y_s \in U(s, x_s)$ *such that the solution to the above system of differential equations starting at* $(x_s, u(s, x(s)))$ *at time* s *satisfies*

$$\forall\, \tau \in [s, t], \;\; y(\tau) \;\in\; U(\tau, x(\tau)), \;\; x(t) \;=\; x \;\&\; y(t) = y$$

If we assume furthermore that the set-valued map Φ *defines an invariant manifold under* (f, g), *then* U *is the* **unique** *set-valued map satisfying the above conditions, which also defines an invariant manifold under* (f, g).

Proof — By assumption, $\mathrm{Graph}(\Psi)$ is contained in $\mathrm{Graph}(\Phi)$. The graphs of the maps Φ and Ψ are closed subsets and repellers since they are contained in $\mathbf{R}_+ \times K \times \mathbf{R}^p$, which is obviously a repeller under the map $-h$.

Therefore, by Theorem 2.4.1, $\mathrm{Graph}(U) := \mathrm{Capt}_{-h}^{\mathrm{Graph}(\Phi)}(\Psi)$ is the largest closed subset $D := \mathrm{Graph}(U)$ satisfying

$$\begin{cases} i) & \mathrm{Graph}(\Psi) \;\subset\; \mathrm{Graph}(U) \;\subset\; \mathrm{Graph}(\Phi) \\ ii) & \mathrm{Graph}(U) \backslash \mathrm{Graph}(\Psi) \;\text{ is locally viable under } \; -h \end{cases}$$

which are translated into the first properties of the above theorem. If we assume furthermore that Φ defines an invariant manifold under (f, g), Theorem 2.5.1 implies that $\mathrm{Graph}(U) = \mathrm{Capt}_{-h}(\Psi)$ is the unique closed subset satisfying the above properties and being backward invariant under $-h$, i.e., invariant under h. $\square$

Theorem 2.4.1 and 2.5.1 also allow us to derive the existence and the uniqueness of a set-valued map solution to the system of first-order partial differential equations (5.2)

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}u(t,x)f(t,x,u(t,x)) - g(t,x,u(t,x)) \ = \ 0$$

satisfying the conditions (5.4)

$$\forall\, (t,x) \in \mathbf{R}_+ \times K, \ \ \Psi(t,x) \ \subset \ U(t,x) \ \subset \ \Phi(t,x)$$

Definition 5.2.4 *We say that a set-valued map $U : \mathbf{R}_+ \times K \mapsto \mathbf{R}^p$ is a Frankowska solution to the problem ((5.2),(5.4)) if the graph of U is closed and if*

$$\left\{ \begin{array}{ll} i) & \forall\, (t,x) \in \mathbf{R}_+ \times K, \ \forall\, y \in U(t,x)\backslash\Psi(t,x), \\ & 0 \ \in \ DU(t,x,y)(-1,-f(t,x,y)) + g(t,x,y) \\ & \text{and} \\ ii) & \forall\, (t,x) \in \mathbf{R}_+ \times K, \ \forall\, y \in U(t,x), \ 0 \ \in \ DU(t,x,y)(1,f(t,x,y)) - g(t,x,y) \end{array} \right.$$

Naturally, if $\Psi := \psi$ and $U := u$ are single-valued on their domains, then a Frankowska solution can be written

$$\left\{ \begin{array}{l} \forall\, (t,x) \in \mathrm{Dom}(u)\backslash\mathrm{Dom}(\psi), \ 0 \in Du(t,x)(-1,-f(t,x,u(t,x))) + g(t,x,u(t,x)) \\ \text{and} \\ \forall\, t \geq 0, \, \forall\, x \in K, 0 \ \in \ Du(t,x)(1,f(t,x,u(t,x)) - g(t,x,u(t,x)) \end{array} \right.$$

If $Du(t,x)(-1,-\xi) = -Du(t,x)(1,\xi)$, these two equations boil down to only one of them on $\mathrm{Dom}(u)\backslash\mathrm{Dom}(\psi)$. If u is differentiable in the usual sense, it satisfies the above first order partial differential equation in the usual sense outside the domain of ψ.

Theorem 5.2.5 *Let us assume that the maps f and g are continuous with linear growth and that the graphs of the set-valued maps $\Psi \subset \Phi$ are closed.*

Then the set-valued map U defined by (5.6) is the **largest** *set-valued map with closed graph satisfying the condition*

$$\forall\, (t,x) \in \mathbf{R}_+ \times K, \ \ \Psi(t,x) \ \subset \ U(t,x) \ \subset \ \Phi(t,x)$$

and solution to

$$\forall\, (t,x) \in \mathbf{R}_+ \times K, \ \forall\, y \in U(t,x)\backslash\Psi(t,x), \ 0 \ \in \ DU(t,x,y)(-1,-f(t,x,y)) + g(t,x,y)$$

If we assume furthermore that f and g are uniformly Lipschitz with respect to x and y and that Φ defines an invariant manifold under (f,g), then U is the **unique** *Frankowska solution to the problem ((5.2),(5.4)).*

Proof — By Theorem 2.4.1, $\text{Graph}(U) := \text{Capt}_{-h}^{\text{Graph}(\Phi)}(\text{Graph}(\Psi))$ is the largest closed subset $D := \text{Graph}(U)$ satisfying

$$\begin{cases} i) & \text{Graph}(\Psi) \subset \text{Graph}(U) \subset \text{Graph}(\Phi) \\ ii) & \forall\,(t,x,y) \in \text{Graph}(U)\backslash\text{Graph}(\Psi), \\ & -(1, f(t,x,t), g(t,x,t)) \in T_{\text{Graph}(U)}(t,x,y) = \text{Graph}(DU)(t,x,y) \end{cases}$$

which can be translated

$$\forall\,(t,x),\ \ \forall\, y \in U(t,x)\backslash\Psi(t,x),\ \ -g(t,x,y) \in DU(t,x,y)(-1,-f(t,x,y))$$

If we assume furthermore that Φ defines an invariant manifold under (f,g), Theorem 2.5.1 implies that $\text{Graph}(U) = \text{Capt}_{-h}(\text{Graph}(\Psi))$ is the unique closed subset satisfying the above properties and being backward invariant under $-h$, i.e., invariant under h. This can be translated by stating that

$$\begin{cases} \forall\,(t,x,y) \in \text{Graph}(U), \\ (1, f(t,x,t), g(t,x,t)) \in T_{\text{Graph}(U)}(t,x,y) = \text{Graph}(DU)(t,x,y) \end{cases}$$

i.e.,

$$\forall\,(t,x) \in \mathbf{R}_+ \times K,\ \forall\, y \in U(t,x),\ \ g(t,x,y) \in DU(t,x,y)(1, f(t,x,y))$$

5.3 Single-Valued Frankowska Solutions

We already mentioned that even when Ψ is single-valued on its domain, the solution U can take several values, defined as "shocks" in the language of physicists.

However, single-valuedness is naturally preserved whenever

$$h(\tau,x,y) := (1, \varphi(x), g(\tau,x,y))$$

when the second component of the map h does not depend upon the second variable y and the differential equation $(\tau', x', y') = h(\tau,x,y)$ has a unique solution for any initial condition.

Therefore, we proceed with the specific case when $f(t,x,y) \equiv \varphi(x)$ depends only upon the variable x.

When $(t,x) \in \mathbf{R}_+ \times K$ is chosen, we introduce the function $x(\cdot) := \vartheta_\varphi(\cdot - t, x)$ the solution to the differential equation $x' = \varphi(x)$ starting at time 0 at $\vartheta_\varphi(-t,x)$, or arriving at x at time t. We associate with it the map $g_{(t,x)} : \mathbf{R}_+ \times \mathbf{R}^p \mapsto \mathbf{R}^p$ defined by

$$\forall\,\tau \geq 0,\ y \in \mathbf{R}^p,\ \ g_{(t,x)}(\tau,y) := g(\tau, \vartheta_\varphi(\tau - t, x), y)$$

We denote by $\vartheta_{g_{(t,x)}}(t,s,y(s))$ the value at t of the solution to the differential equation

$$y'(\tau) \;=\; g_{(t,x)}(\tau,y(\tau)) \;:=\; g(\tau,\vartheta_{\varphi}(\tau-t,x),y(\tau))$$

starting at $y(s)$ associated with the evolution $x(\tau) := \vartheta_{\varphi}(\tau-t,x)$ starting at $x(s) = \vartheta_{\varphi}(s-t,x)$ at initial time s.

We associate with the backward exit function the map $\Theta_K^{-\varphi}$ defined by

$$\forall\, x \in K,\;\; \Theta_K^{-\varphi}(x) \;:=\; \vartheta_{-\varphi}(\tau_K^{-\varphi}(x),x)$$

and we say that $\Theta_K^{-\varphi}$ is the "exitor" (for exit projector) of K. It maps K to its boundary Γ and satisfies $\Theta_K^{-\varphi}(x) = x$ for every $x \in \Theta_K^{-\varphi}(K)$.

It will be very convenient to extend the function $\tau_K^{-\varphi}$ defined on K to the function (again denoted by) $\tau_K^{-\varphi}$ defined on $\mathbf{R}_+ \times K$ by $\tau_K^{-\varphi}(t,x) := \min(t,\tau_K^{-\varphi}(x))$:

$$\tau_K^{-\varphi}(t,x) \;:=\; \begin{cases} t & \text{if} \;\; t \in [0,\tau_K^{-\varphi}(x)] \\ \tau_K^{-\varphi}(x) & \text{if} \;\; t \in]\tau_K^{-\varphi}(x),\infty[\end{cases}$$

so that we can also extend the exitor map by setting

$$\Theta_K^{-\varphi}(t,x) \;:=\; \vartheta_{-\varphi}(\tau_K^{-\varphi}(t,x),x) \;\in\; K$$

because we observe that $\Theta_K^{-\varphi}(t,x)$ is equal to

$$\begin{cases} \vartheta_{-\varphi}(t,x) & \text{if} \;\; t \in [0,\tau_K^{-\varphi}(x)] \\ \Theta_K^{-\varphi}(x) & \text{if} \;\; t \in]\tau_K^{-\varphi}(x),\infty[\end{cases}$$

Proposition 5.3.1 *We posit assumption*

$$K \;\text{ is closed and (forward) invariant under }\; \varphi \tag{5.7}$$

and

$$\begin{cases} i) & \varphi \;\text{ is Lipschitz and that } g \text{ is continuous} \\ ii) & (\tau',x',y') = h(\tau,x,y) \text{ has a unique solution for any initial condition} \end{cases} \tag{5.8}$$

Let us introduce

1. *an* initial data $u_0 : K \mapsto \mathbf{R}^p$,

2. *a* boundary data $v_\Gamma : \mathbf{R}_+ \times \partial K \mapsto \mathbf{R}^p$

The solution $u := \mathcal{A}_{(\varphi,g)}(\Psi(u_0, v_\Gamma))$ *is the single-valued map with closed graph defined by*

$$u(t,x) \;=\; \vartheta_{g_{(t,x)}}(t, t-\tau_K^{-\varphi}(t,x), (\Psi(u_0,v_\Gamma)(t-\tau_K^{-\varphi}(t,x), \Theta_K^{-\varphi}(t,x))) \tag{5.9}$$

or, more explicitly, by

$$\begin{cases} \vartheta_{g_{(t,x)}}(t, 0, u_0(\vartheta_{-\varphi}(t,x))) & \text{if} \quad t \in [0, \tau_K^{-\varphi}(x)] \\ \\ \vartheta_{g_{(t,x)}}(t, t-\tau_K^{-\varphi}(x), v_{\partial K}(t-\tau_K^{-\varphi}(x), \Theta_K^{-\varphi}(x))) & \text{if} \quad t \in]\tau_K^{-\varphi}(x), \infty[\end{cases}$$

Furthermore, if we assume the following viability assumptions on Φ

$$\begin{cases} i) & \forall\, x \in K,\ \forall\, t \geq 0,\ \ \forall\, y \in \Phi(t,x),\ \ g(t,x,y) \;\in\; D\Phi(t,x,y)(1,\varphi(x)) \\ ii) & \forall\, \xi \in \Gamma,\ \forall\, t > 0,\ \ v_\Gamma(t,\xi) \;\in\; \Phi(t,\xi) \\ iii) & \forall\, x \in K,\ \ u_0(x) \;\in\; \Phi(0,x) \end{cases} \tag{5.10}$$

then

$$\forall\, (t,x) \in \mathbf{R}_+ \times K,\ \ u(t,x) \;\in\; \Phi(t,x)$$

Remark: — We stress the fact that the solution $u(t,x)$ depends only

1. upon the initial condition $u_0(x)$ when $t \leq \tau_K^{-\varphi}(x)$,
2. upon the boundary condition v_Γ when $t > \tau_K^{-\varphi}(x)$.

The second property proves a general principle concerning demographic evolution stating the state of the system eventually forgets its initial condition $u_0(\cdot)$.

Proof — We first take $\Phi(t,x) \equiv \mathbf{R}^p$, which is invariant by assumption (5.7). Then the graph of u is defined by (5.6) is equal to

$$\text{Graph}(U) \;:=\; \text{Capt}_{-h}(\text{Graph}(\Psi(u_0, v_\Gamma)))$$

An element (t,x,y) of the graph of U is the value at some $h \geq 0$ of the solution $(\tau(\cdot), x(\cdot), y(\cdot))$ to the system of differential equations

$$\begin{cases} i) & \tau'(t) \;=\; 1 \\ ii) & x'(t) \;=\; \varphi(x(t)) \\ iii) & y'(t) \;=\; g(\tau(t), x(t), y(t)) \end{cases}$$

starting from $(s, c, \Psi(u_0, v_\Gamma)(s,c)))$, assumed to be unique.

This implies that $t = s + h$ and $x = \vartheta_\varphi(t, c)$. If $s = 0$, then $t = h$, $x = \vartheta_\varphi(t, c)$ and

$$y \;=\; \vartheta_{g_{(t,x)}}(t, 0, u(0, \vartheta_{-\varphi}(t, x)))$$

If $s > 0$, then $c \in \Gamma$, so that $h = \tau_K^{-\varphi}(x)$, $s = t - \tau_K^{-\varphi}(x)$ and $c = \Theta_K^{-\varphi}(x)$ and

$$y \;=\; \vartheta_{g_{(t,x)}}(t, t - \tau_K^{-\varphi}(x), v_\Gamma(t - \tau_K^{-\varphi}(x), \Theta_K^{-\varphi}(x)))$$

Therefore, $y =: u(t, x)$ is uniquely determined by t and x so that $U =: u$ is single-valued.

Assumptions (5.10) imply that the graph of the map $(t, x) \rightsquigarrow \Phi(t, x)$ is invariant under $(1, \varphi, g)$ and that $\mathrm{Graph}(\Psi_{u_0, v_\Gamma})$ is contained in the graph of Φ. Hence the graph of u is contained in the graph of Φ. $\square$

Example Let us consider a x-dependent matrix $A(x) \in \mathcal{L}(\mathbf{R}^p, \mathbf{R}^p)$. We associate the differential equation

$$y'(\tau) \;=\; -A(x(\tau))y(\tau)$$

Then $u(t, x)$ is equal to

$$\begin{cases} e^{-\int_0^t A(\vartheta_\varphi(\tau - t, x)d\tau} u_0(\vartheta_{-\varphi}(t, x)) & \text{if } \; t \in [0, \tau_K^{-\varphi}(x)] \\ e^{-\int_{t-\tau_K^{-\varphi}(x)}^t A(\vartheta_\varphi(\tau - t, x)d\tau} v_\Gamma(t - \tau_K^{-\varphi}(x), \Theta_K^{-\varphi}(x)) & \text{if } \; t \in]\tau_K^{-\varphi}(x), \infty[\end{cases}$$

Example Let us consider the case when $K \;:=\; K_1 \times \cdots \times K_n$ here the $K_i \subset \mathbf{R}_i^n$ are close subsets and set $\mathbf{R}^n := \prod_{j=1}^n \mathbf{R}_j^n$.

Lemma 5.3.2 *Assume that* $K := \prod_{j=1}^n K_j$ *is the product of* n *closed subsets* $K_j \subset \mathbf{R}_j^n$. *We posit assumptions (5.7) on* K. *Denoting by*

$$\tau_{K_j}^{-\varphi}(x) \;:=\; \tau_{K_j}((\vartheta_{-\varphi}(\cdot, x))_j)$$

the partial backward exit time functions, *the backward exit time function can then be written*

$$\tau_K^{-\varphi}(x) \;:=\; \min_{j=1,\ldots,n} \tau_{K_j}^{-\varphi}(x)$$

Proof — We observe that

$$\prod_{j=1}^n \mathbf{R}_j^n \backslash \left(\prod_{j=1}^n K_j \right) \;=\; \bigcup_{j=1}^n \left(\prod_{i=1}^{j-1} \mathbf{R}_i^n \times (\mathbf{R}_j^n \backslash K_j) \times \prod_{l=j+1}^n \mathbf{R}_l^n \right)$$

and thus, that for any function $t \mapsto x(t) = (x_1(t), \ldots, x_n(t))$,

$$\tau_K(x(\cdot)) \ := \inf_{x(t)\in\mathbf{R}^n\setminus K} t \ = \min_{j=1,\ldots,n}\left(\inf_{x_j(t)\in\mathbf{R}^n_j\setminus K_j} t\right) \ = \min_{j=1,\ldots,n}\tau_{K_j}(x_j(\cdot))$$

since the infimum on a finite union of subsets is the minimum of the infima on each subsets. $\square$

In this case,

$$\partial\left(\prod_{j=1}^{n} K_j\right) \ = \ \bigcup_{j=1}^{n}\left(\prod_{i=1}^{j-1} K_i \times \Gamma_j \times \prod_{l=j+1}^{n} K_l\right)$$

so that the boundary data defined on Γ are defined by n maps

$$v_\Gamma^j : \prod_{i=1}^{j-1} K_i \times \Gamma_j \times \prod_{l=j+1}^{n} K_l \mapsto \mathbf{R}^p$$

Example For instance, let us consider the case when the four-dimensional causal variable $x := (x_1, x_2, x_3, x_4)$ ranges over the product $K := \prod_{i=1}^4 K_i$ with $K_1 := \mathbf{R}_+$, $K_2 := [0, r_2]$, $K_3 := \mathbf{R}_+$ and $K_4 := [0, b]$.

We are looking for solutions to the system of first-order partial differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) + \dfrac{\partial}{\partial x_1}u(t,x) - \rho\dfrac{\partial}{\partial x_2}u(t,x) + \sigma\dfrac{\partial}{\partial x_3}u(t,x) + \beta(b-x_4)x_4\dfrac{\partial}{\partial x_4}u(t,x) \\ = \ g(t,x,u(t,x)) \end{cases}$$

(where the scalars functions ρ, σ and β are positive) satisfying the initial and boundary conditions

$$\begin{cases} i) & \forall\, x_1 \geq 0,\ \forall\, x_2 \in [0, r_2],\ x_3 \geq 0,\ x_4 \in [0,b], \\ & u(0, x_1, x_2, x_3, x_4) \ = \ u_0(x_1, x_2, x_3, x_4) \\ ii) & \forall\, t \geq 0,\ \forall\, x_2 \in [0, r_2],\ x_3 \geq 0,\ x_4 \in [0,b], \\ & u(t, 0, x_2, x_3, x_4) \ = \ v_1(t, x_2, x_3, x_4) \\ i) & \forall\, t \geq 0,\ \forall\, x_1 \geq 0,\ x_3 \geq 0,\ x_4 \in [0,b], \\ & u(t, x_1, r_2, x_3, x_4) \ = \ v_{r_2}(t, x_1, x_3, x_4) \end{cases}$$

Hence, we derive the existence and the uniqueness of the Frankowska solution to the above system of partial differential equations satisfying an initial condition, a boundary condition for $x_1 = 0$ (births) and a boundary condition for $x_2 = r_2$.

For computing it, we need to know the backward exit time and the exitor for the associated characteristic system given by

$$\left\{\begin{array}{lrcl} i) & x_1'(t) & = & 1 \\ ii) & x_2'(t) & = & -\rho x_2(t) \\ iii) & x_3'(t) & = & \sigma x_3(t) \\ iv) & x_4'(t) & = & \beta(b - x_4(t))x_4(t) \end{array}\right. \tag{5.11}$$

where $\varphi(x) := (\varphi_i(x))_{i=1,\ldots,4}$ with $\varphi_1(x_1) := 1$, $\varphi_2(x_2) := -\rho x_2$, $\varphi_3(x_3) := \sigma x_3$ and $\varphi_4(x_4) := \beta(b - x_4)x_4$.

We recall that the solution to the purely logistic equation $y'(t) = \beta(t)(b - y(t))y(t)$ starting at y_s at time s is given by

$$y(t) \;:=\; \frac{b}{1 + \left(\frac{b}{y_s} - 1\right) e^{-b\int_s^t \beta(\tau)d\tau}}$$

The closed subset K is obviously (forward) invariant under φ defined by φ and one can observe easily that

$$\tau_{K_3}^{-\varphi}(x) \;=\; +\infty \;\&\; \tau_{K_4}^{-\varphi}(x) \;=\; +\infty$$

so that

$$\tau_K^{-\varphi}(t, x_1, x_2, x_3, x_4) \;=\; \min\left(t, x_1, \frac{1}{\rho}\log\left(\frac{r_2}{x_2}\right)\right)$$

Therefore,

1. if $t \leq \min\left(x_1, \frac{1}{\rho}\log\left(\frac{r_2}{x_2}\right)\right)$, then

$$\Theta_K^{-\varphi}(t,x) \;=\; \left(0, x_1 - t, e^{\rho t}x_2, e^{-\sigma t}x_3, \frac{b}{1 + \left(\frac{b}{x_4} - 1\right)e^{\beta b t}}\right)$$

2. if $x_1 \leq \min\left(t, \frac{1}{\rho}\log\left(\frac{r_2}{x_2}\right)\right)$, then

$$\Theta_K^{-\varphi}(t,x) \;=\; \left(t - x_1, 0, e^{\rho x_1}x_2, e^{-\sigma x_1}x_3, \frac{b}{1 + \left(\frac{b}{x_4} - 1\right)e^{\beta b x_1}}\right)$$

3. if $\frac{1}{\rho}\log\left(\frac{r_2}{x_2}\right) \leq \min(t, x_1)$, then

$$\left\{\begin{array}{l} \Theta_K^{-\varphi}(t,x) \;= \\ \left(t - \dfrac{1}{\rho}\log\left(\dfrac{r_2}{x_2}\right), x_1 - \dfrac{1}{\rho}\log\left(\dfrac{r_2}{x_2}\right), r_2, e^{\left(\frac{x_2}{r_2}\right)^{\frac{\sigma}{\rho}}} x_3, \dfrac{b}{1 + \left(\frac{b}{x_4} - 1\right)\left(\frac{r_2}{x_2}\right)^{\frac{\beta b}{\rho}}}\right) \end{array}\right.$$

If the right-hand side $g(t,x,y) := -A(t,x)y$ where $A(t,x) \in \mathcal{L}(\mathbf{R}^p, \mathbf{R}^p)$ is linear, then we set

$$A(t,x;\tau) \;:=\; A\left(\tau, x_1 - t + \tau, e^{-\rho(t-\tau)}x_2, e^{\sigma(t-\tau)}x_3, \frac{b}{1+\left(\frac{b}{x_4}-1\right)e^{\beta b(t-\tau)}}\right)$$

Then the solution is given by

1. if $t \leq \min\left(x_1, \frac{1}{\rho}\log\left(\frac{r_2}{x_2}\right)\right)$, then

$$u(t,x) \;=\; e^{-\int_0^t A(t,x;\tau)d\tau} u_0\left(x_1 - t, e^{\rho t}x_2, e^{-\sigma t}x_3, \frac{b}{1+\left(\frac{b}{x_4}-1\right)e^{\beta bt}}\right)$$

2. if $x_1 \leq \min\left(t, \frac{1}{\rho}\log\left(\frac{r_2}{x_2}\right)\right)$, then

$$u(t,x) \;=\; e^{-\int_{t-x_1}^t A(t,x;\tau)d\tau} v_1\left(t - x_1, e^{\rho x_1}x_2, e^{-\sigma x_1}x_3, \frac{b}{1+\left(\frac{b}{x_4}-1\right)e^{\beta b x_1}}\right)$$

3. if $\frac{1}{\rho}\log\left(\frac{r_2}{x_2}\right) \leq \min(t, x_1)$, then

$$\begin{cases} u(t,x) \;=\; e^{-\int_{t-\frac{1}{\rho}\log\left(\frac{r_2}{x_2}\right)}^t A(t,x;\tau)d\tau} \\ v_{r_2}\left(t - \dfrac{1}{\rho}\log\left(\dfrac{r_2}{x_2}\right), x_1 - \dfrac{1}{\rho}\log\left(\dfrac{r_2}{x_2}\right), e^{\left(\frac{x_2}{r_2}\right)^{\frac{\sigma}{\rho}}}x_3, \dfrac{b}{1+\left(\frac{b}{x_4}-1\right)\left(\frac{r_2}{x_2}\right)^{\frac{\beta b}{\rho}}}\right) \end{cases}$$

5.4 Regularity Properties

We begin by proving that the operator $\mathcal{A}_{(\varphi,g)}$ preserves continuity and boundedness:

Proposition 5.4.1 *We posit assumptions (5.7) and (5.8) and the regularity property*

$$\tau_K^{-\varphi} : K \mapsto \Gamma \text{ is continuous} \tag{5.12}$$

Assume also that g enjoys uniform linear growth with respect to y in the sense that there exists a positive constant c such that

$$\forall\, x \in K,\ \forall\, y \in \mathbf{R}^p,\ \ \|g(t,x,y)\| \;\leq\; c(1 + \|y\|) \tag{5.13}$$

Then the $\mathcal{A}_{(\varphi,g)}(\Psi(u_0, v_\Gamma))$ is continuous and bounded whenever u_0 and v_Γ are continuous and bounded.

Proof — By assumption (5.12), $\tau_K(t,x) := \min(t, \tau_K^{-\varphi}(x))$ is also continuous, so that $\Theta_K^{-\varphi}(t,x) := \vartheta_{-\varphi}(\tau_K^{-\varphi}(t,x), x)$ is also continuous, and thus

$$u(t,x) \;=\; \vartheta_{g_{(t,x)}}(t, t-\tau_K(t,x), \Psi(u_0, v_\Gamma)(t-\tau_K(t,x), \Theta_K^{-\varphi}(t,x)))$$

is also continuous. When $\|g(t,x,y)\| \;\leq\; c(1+\|y\|)$, we also infer that

$$\|u(t,x)\| \;\leq\; e^{c\tau_K^{-\varphi}(t,x)} \max(\|u_0(x)\|, \|v_\Gamma(\Theta_K^{-\varphi}(x))\|)$$

and thus, that

$$\|u(t,\cdot)\|_\infty \;\leq\; e^{ct} \max(\|u_0\|_\infty, \|v_\Gamma\|_\infty)$$

Next, we prove that under monotonicity assumptions, the operator $\mathcal{A}_{(\varphi,g)}$ is Lipschitz:

Definition 5.4.2 *We say that g is* uniformly monotone *with respect to t and y if there exists $\mu \in \mathbf{R}$ such that*

$$\langle g(t,x,y_1) - g(t,x,y_2), y_1 - y_2\rangle \;\leq\; -\mu\|y_1 - y_2\|^2 \tag{5.14}$$

The interesting case is obtained when $\mu > 0$. When g is uniformly Lipschitz with respect to t and y with constant λ, then it is uniformly monotone with $\mu = -\lambda$.

Proposition 5.4.3 *We posit assumption (5.7) and assume that φ is Lipschitz and that g is continuous and that g is* uniformly monotone *with respect to y.*

Then, for each $t > 0$,

$$\left\{ \begin{array}{l} \sup_{x\in K} \|(\mathcal{A}_{(\varphi,g)}\Psi(u_0^1, v_\Gamma^1))((t,x) - (\mathcal{A}_{(\varphi,g)}\Psi(u_0^2, v_\Gamma^2)(t,x)\| \\ \\ \leq\; e^{-\mu t}\max(\|u_0^1 - u_2^2\|_\infty, \|v_\Gamma^1(t,\cdot) - v_\Gamma^2)(t,\cdot)\|_\infty) \end{array} \right.$$

Consequently, if we posit assumptions (5.12) and (5.13), we deduce that for any $T > 0$

$$\mathcal{A}_{(\varphi,g)} : \mathcal{C}_\infty(K, \mathbf{R}^p) \times \mathcal{C}_\infty([0,T]\times\Gamma, \mathbf{R}^p) \mapsto \mathcal{C}_\infty([0,T]\times K, \mathbf{R}^p)$$

is a Lipschitz operator from the space $\mathcal{C}_\infty(K, \mathbf{R}^p) \times \mathcal{C}_\infty([0,T]\times\Gamma, \mathbf{R}^p)$ of pairs of continuous and bounded initial and boundary data to the space $\mathcal{C}_\infty([0,T]\times K, \mathbf{R}^p)$ of continuous and bounded maps from $[0,T]\times K$ to $\mathbf{R}^p$.

Proof — Indeed, setting $u^i(t,x) := (\mathcal{A}_{(\varphi,g)}(\Psi(u_0^i, v_\Gamma^i)))(x)$, $i = 1,2$, we know that

$$\forall\, x \in K,\;\; u^i(t,x) \;:=\; \vartheta_{g_{t,x}}(t, t-\tau_K^{-\varphi}(t,x), \Psi(u_0^i, v_\Gamma^i)(\Theta_K^{-\varphi}(t,x)))$$

is the value $y^i(t) := u^i(t,x)$ at time t of the solution $y(\cdot)$ to the differential equation $y'(\tau) = g(\tau, x(\tau), y(\tau))$ starting from $\Psi(u_0^i, v_\Gamma^i)(\Theta_K^{-\varphi}(t,x))$ at time $t-\tau_K^{-\varphi}(t,x)$, where $x(\cdot)$ is the solution to the differential equation $x' = \varphi(x)$ starting from $\Theta_K^{-\varphi}(t,x)$ at time $t - \tau_K^{-\varphi}(t,x)$.

Recalling that $\tau_K^{-\varphi}(t,x) = \min(t, \tau_K^{-\varphi}(x)) \leq t$, we infer from assumption (5.14) that[1]

$$\begin{cases} \|y^1(t) - y^2(t)\| \\ \leq\ e^{-\mu\tau_K^{-\varphi}(t,x)} \|\Psi(u_0^1, v_\Gamma^1)(\Theta_K^{-\varphi}(t,x)) - \Psi(u_0^2, v_\Gamma^2)(\Theta_K^{-\varphi}(t,x))\| \\ \leq\ e^{-\mu t} \max(\sup_{x\in K} \|u_0^1(x) - u_0^2(x)\|, \sup_{\xi\in\Gamma} \|v_\Gamma^1(\xi) - v_{\partial K}^2(\xi)\|) \\ =\ e^{-\mu t} \max(\|u_0^1 - u_0^2\|_\infty + \|v_\Gamma^1 - v_\Gamma^2\|_\infty) \end{cases}$$

[1]Indeed, integrating the two sides of inequality

$$\frac{d}{dt}\|y_1(t) - y_2(t)\|^2 \ =\ 2\langle g(t,x(t),y_1(t)) - g(t,x(t),y_2(t)), y_1(t) - y_2(t)\rangle \ \leq\ -2\mu\|y_1(t) - y_2(t)\|^2$$

yields

$$\|y_1(t) - y_2(t)\|^2 \ \leq\ e^{-2\mu t}\|y_1(0) - y_2(0)\|^2$$

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